RAMReS Sciences des Structures et de la Matière Vol. 9, N°2 (2025) 14 - 21
http://publication.lecames.org/index.php/mat *ISSN: 2630-1180*

Privileged Complexity of the ternary Thue-Morse word

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Received: 05 May 2025 / Received in revised form: 28 September 2025 / Accepted: 12 December 2025

***Abstract*:**
In this paper, we study a new type of complexity for infinite words, called *privileged complexity*. We apply this concept to the ternary Thue-Morse word. First, we present properties of the return words in the ternary Thue-Morse word. Then, we study the privileged words of this infinite word. Finally, we derive a recursive formula that allows us to determine the privileged complexity function of this ternary word.
2020 Mathematics Subject Classification: 68R15, 11B85.

***Keywords*:** Infinite words; First complete return word; Privileged word; Privileged complexity.

1. Introduction

The Thue-Morse word $\mathbf{t}_2$ is the fixed point of the morphism μ defined over $\{0,1\}$ by $\mu(0) = 01$ and $\mu(1) = 10$. This word has been extensively studied in combinatorics on words and has been used to solve several problems in number theory (see [1–5]). In an infinite overlap-free word $\mathbf{u}$, the return word of a factor v that appears at least twice in $\mathbf{u}$ is the word w such that wv is a factor of $\mathbf{u}$ starting with v and containing exactly two occurrences of v. In this case, the word wv is called a *first complete return word* in $\mathbf{u}$. Privileged words are defined recursively as follows: the empty word and every letter of a given alphabet are privileged. A finite nonempty word v of length at least 2 is privileged if and only if it is the first complete return of

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https://doi.org/10.70974/mat0922514

a nonempty privileged word.

The privileged complexity function of an infinite word $\mathbf{u}$ counts the number of distinct privileged words of a given length in $\mathbf{u}$. In [6], the authors studied the palindromic factorization of words and established a connection to privileged words. In [7, 8], it is shown that, in a rich finite or infinite word, each nonempty factor w is a palindrome if and only if w is privileged. The notion of a first complete return was used to characterize palindromic factors, which made it possible to show that the set of palindromic factors is the same as the set of privileged words in a rich, infinite word (see [9]).

A recursive formula describing the privileged complexity function of the Thue-Morse word was established in [10]. The author shows that this function is unbounded and that its values contain arbitrarily long sequences of zeros.

A natural generalization of the Thue-Morse word over a three-letter alphabet was introduced in [11]. This is the word $\mathbf{t}_3$ generated over $\{0, 1, 2\}$ by the morphism μ_3, which is formalized as follows:

$$\mu_3(0) = 012, \ \mu_3(1) = 120 \text{ and } \mu_3(2) = 201.$$

In [3], the authors studied the separator factors of $\mathbf{t}_3$ and showed that any factor of this word has 7, 8 or 9 return words. The recurrence function of $\mathbf{t}_3$ was also studied in [12].

Our goal is to generalize the results on the privileged complexity function of $\mathbf{t}_2$ from [10] to $\mathbf{t}_3$. Thus, the main contribution of this paper is a complete recursive formula for computing this complexity for $\mathbf{t}_3$.

This paper is organized as follows: First, in Section 2, we provide some useful definitions and notation. Section 3 provides properties of the privileged words of $\mathbf{t}_3$. Section 4 concludes by establishing a recursive function for computing the privileged complexity of $\mathbf{t}_3$.

2. Definitions and notation

An alphabet $\mathcal{A}$ is a finite set of symbols whose elements are called *letters*. A *finite word* over $\mathcal{A}$ is a sequence of letters. To the empty sequence corresponds the *empty word*, denoted by ε. The set of all finite words over $\mathcal{A}$ is denoted by $\mathcal{A}^*$. The set of non-empty words over $\mathcal{A}$ is the set $\mathcal{A}^+ = \mathcal{A}^* - \{\varepsilon\}$. A natural binary operation on words is concatenation. Thus, under this operation, $\mathcal{A}^*$ is a free monoid over $\mathcal{A}$. For a given finite word u, the length of u is denoted $|u|$. An infinite word is sequence indexed by the natural numbers with values in $\mathcal{A}$. We write concisely $\mathbf{u} = a_1a_2a_3\cdots$ with $a_i \in \mathcal{A}$. The set of infinite words over $\mathcal{A}$ is denoted $\mathcal{A}^\omega$.

A finite word v is a factor of the finite or infinite word u if there are $w \in \mathcal{A}^*$ and $z \in \mathcal{A}^* \cup \mathcal{A}^\omega$ such that $u = wvz$. If $w = \varepsilon$ (resp. $z = \varepsilon$) , then v is a prefix of u (resp. suffix of u). Let u, v_1, v_2, and w be finite words such that $u = v_1wv_2$. We say that w is *median factor* of u if $|v_1|= |v_2|$.
Let v be a factor of u and $a \in \mathcal{A}$. We say that v is right (resp. left) prolongable by a, if va (resp. av) is still a factor of u. The factor v is said to be right (resp. left) special if it admits at least two right (resp. left) extensions. In this case, va (resp. av) is called a right (resp. left) extension of v in u. It is called bispecial if it is both right and left special. An infinite word $\mathbf{u}$ is said recurrent if each of its factors appears an infinitly times. When a finite word v begins (resp. ends) with a letter a, we define $a^{-1}v$ (resp. va^{-1}) as the deletion of the first (resp. last) letter of v.

A morphism f is a mapping of $\mathcal{A}^*$ in itself such that $f(uv) = f(u)f(v)$, for all $u, v \in \mathcal{A}^*$. The morphism f is said to be primitive if there exists an integer n such that, for any letter a of $\mathcal{A}$, $f^n(a)$ contains all the letters of $\mathcal{A}$. It is said to be k-uniform, if $|f(a)| = k$ for all $a \in \mathcal{A}$. We say that f is prolongable in $a \in \mathcal{A}$ if $f(a) = aw$ where $w \in \mathcal{A}^+$, and $f^n(w)$ is non-empty for any integer n. We say that f is left-marked over $\mathcal{A}$ if all its images start with different letters. The notion of right-marked is defined in a similar way. When f is marked both on left and right, it is said to be simply marked. An infinite word $\mathbf{u}$ is generated by a morphism f if there exists a letter a of $\mathcal{A}$ such that the words $a, f(a)\cdots, f^n(a), \cdots$ are increasingly long prefixes of $\mathbf{u}$. In this case, we note $\mathbf{u} = f^{\omega}(a)$. An infinite word generated by a morphism is called a purely morphic word. We define the exchange morphism E_3 over $\mathcal{A}_3 = \{0,\ 1\ 2\}$ by:

$$E_3(0) = 1,\ E_3(1) = 2,\ E_3(2) = 0.$$

Given an infinite word $\mathbf{w}$, let $F_{\mathbf{w}}(n)$ (resp. $F_{\mathbf{w}}$) denote the set of factors of length n (resp. all factors) in $\mathbf{w}$. A factor v that separates two squares of letters in $\mathbf{w}$ is called a separator factor of $\mathbf{w}$. If v separates two different squares of letters, such as ii and jj, then it is denoted as v_{ij}. Otherwise, it is denoted as v_{ii}.

Definition 1. (i) An *overlap* is any finite word of the form $avava$, where a is a letter and v is a factor possibly empty.

(ii) A finite word w is *overlap-free* if it contains no overlaps as factors.

Example 1. Consider the alphabet $A = \{a, b, c\}$. Then

1. *acbacba* is an overlap with $v = bc$.
2. *cbacbaa* is overlap-free.

Definition 2. Let $\mathbf{w}$ be a recurrent overlap-free infinite word and u a nonempty factor of $\mathbf{w}$.

(i) A factor v of $\mathbf{w}$ is *a complete first return* of u in $\mathbf{w}$ if it satisfies the following condition:

- v is a prefix of u.
- uv is a factor of $\mathbf{w}$.
- $|uv|_v = 2$.

(ii) A factor v of $\mathbf{w}$ is *a first return* word of u in $\mathbf{w}$ is the shortest word that begins with an occurrence of v and ends just before the next occurrence of v in $\mathbf{w}$.

The set of return words of v is denoted $\mathrm{Ret}(v)$.

Example 2. Consider the alphabet $A_2 = \{0, 1\}$ and the word $w = 1001010010010001$. Then:

1. $v = 10$ is a first complete return of $u = 100$ in w.
2. $v = 1$ is a first return word of in w.
3. $\mathrm{Ret}(1) = \{100, 10, 1000\}$.
4. The set of first complete return words of 1 in w are:

$$\{1001, 101, 10001\}.$$

3. Complete return words and privileged words in $\mathbf{t}_3$

Privileged words are a well-known family of words introduced in [8] and extensively studied for some classical infinite words, such as Sturmian and Thue-Morse words (see [9], [10]). We define the set $Pri_{\mathcal{A}}$ of privileged words over $\mathcal{A}$, recursively as follows:

- $\varepsilon \in Pri_{\mathcal{A}}$,
- $a \in Pri_{\mathcal{A}}$, for every letter a in the alphabet,
- if $|w| \geq 2$, then $w \in Pri_{\mathcal{A}}$ if w is a complete first return to a shorter privileged word.

When the alphabet is known from context, we omit the subscript $\mathcal{A}$. Given an infinite word $\mathbf{w}$, we denote by:

$$Pri_{\mathbf{w}} = \{v \in F_{\mathbf{w}:\ v \text{ is privileged}}\},$$

the set of all privileged words.
The set $Pri_{\mathbf{w}}(n)$ is defined to contain privileged words of $\mathbf{w}$ of length n. Thus, we have:

$$Pri_{\mathbf{w}}(n) = Pri_{\mathbf{w}} \cap F_{\mathbf{w}}(n).$$

The first few binary privileged words are:

$$\varepsilon,\ 0,\ 1,\ 00,\ 11,\ 000,\ 010,\ 101.$$

Not every privileged word is a palindrome; for instance, the words 00101100 and 0120 are privileged but not palindromic. Let v be a privileged word of $\mathbf{w}$ of length n. Then, we denote the set of privileged words starting with v by $Pri_v(n)$. In other words, it is the set of all factors of w that contain exactly two occurrences of v, one as a prefix and the other as a suffix. Thus, we define the privileged complexity function of an infinite word $\mathbf{w}$ as a map from $\mathbb{N}$ to $\mathbb{N}$ by :

$$c_{\mathbf{w}}^{\mathrm{pri}}(n) = \#Pri_{\mathbf{w}}(n), \text{ for all integer } n.$$

Below, we outline some properties of the privileged words of $\mathbf{t}_3$.

Lemma 1. *Let u be a privileged word of $\mathbf{t}_3$ and v a privileged prefix (resp. suffix) of u. Then v is a suffix (resp. prefix) of u.*

Proof. Let v be a privileged prefix of u. If $|u| \leq 1$, the statement is obvious, since v is either the empty word or a single letter. If $u = v$, then v is both a prefix and a suffix of u.

Suppose that $|u| \geq 2$ and $|u| > |v|$. Then, by definition, u is a first complete return of a shorter privileged word w. We proceed by considering $|w|$.

• If $|w| < |v|$: Let v_1 be the smallest privileged word that generates v, and let w_1 be the largest privileged prefix that generates w. We show that w is a suffix of v_1.

If $|w_1| > |v_1|$, then v must contain at least three occurrences of v_1, which is impossible. Therefore, w_1 is a privileged prefix of v_1. Thus, the smallest privileged suffix of v has the largest privileged prefix of w as its privileged prefix. Since v_1 is a privileged word, it is a complete return of another privileged word v_2 with $|v_2| < |w_1|$. Hence, v_2 is a prefix of w_1. It follows that v_1 admits w_1 as a prefix and a privileged prefix of w_1 as a suffix.

By repeating this process with v_2 and so on, we find that there exists a privileged suffix v_k of v_1 such that $k \geq 3$ and v_k contains w_1 as a privileged prefix. This implies that w is a factor of v_1. Since $|w| \geq |v_1|$, we have $w = v_1$; i.e., w is a suffix of v_1.

• If $|w| = |v|$, then v is a suffix of u.

• If $|w| > |v|$, then v is a proper prefix of w. Since w is privileged, v is also a suffix of w and therefore of u.

Lemma 2. *Let u be a privileged word of $\mathbf{t}_3$ and v its longest privileged prefix (resp. suffix). Then u is a first complete return of v.*

Proof. Let u be a privileged word of $\mathbf{t}_3$ and let v be its longest privileged prefix. If $|u| \leq 1$, the statement is obvious.

Now suppose $|u| \geq 2$. Since u is privileged, there exists a privileged word w such that u is a first complete return of w. We consider $|w|$.

If $|v| > |w|$, then w is a prefix of v and, by Lemma 1, also a suffix of v. Consequently, u would contain at least three occurrences of w, contradicting the fact that u is a first complete return of w. Thus, $|v| \leq |w|$.

Since v is the longest privileged prefix of u, we have $|v| = |w|$.

The case where v is the longest suffix of u is treated similarly.

Lemma 3. *Any factor of $\mathbf{t}_3$ is a prefix of a privileged word of $\mathbf{t}_3$.*

Proof. Let v be a factor of $\mathbf{t}_3$. Since $\mathbf{t}_3$ is recurrent, v appears infinitely many times in $\mathbf{t}_3$. Consequently, $\mathbf{t}_3$ contains a factor u of the form $vuvv$. The factor w of $\mathbf{t}_3$ contains neither a prefix nor a suffix of v because $\mathbf{t}_3$ is overlap-free. Therefore, u is a privileged word that starts and ends with v.

Below, we recall a well-known theorem (Theorem 1 in [3]) that provides the number of return words of $\mathbf{t}_3$. The authors prove it using the synchronization lemma (Lemma 4) and stability under the exchange application (Proposition 1) from [3] and [11], respectively.

Theorem 1. *Any factor of $\mathbf{t}_3$ has 7, 8 or 9 return words.*

Lemma 4. *Let u be a factor of $\mathbf{t}_3$. Then there exist factors v, δ_1 and δ_2 of $\mathbf{t}_3$ such that $u = \delta_1\mu_3(v)\delta_2$ with $|\delta_1|, |\delta_2| \leq 2$. Moreover if $|u| \geq 7$ then, this decomposition is unique.*

Proposition 1. *$F_{\mathbf{t}_3}$ is stable by E_3.*

The authors use these results to show that establishing the set of return words requires considering only the factors 0, 01 and 012 from the triprolongable bispecial factor set (three extensions), as well as the factors

$\mu_3(0)0$, $\mu_3(0)01$ from the biprolongable bispecial factor set (two extensions). Thus, we have:

•$\mathrm{Ret}(0) = \{0, 01, 02, 012, 0112, 0122, 01212\}$.
•$\mathrm{Ret}(01) = \{01, 012, 0122, 01212, 01202, 011202, 012120, 0121202\}$.
•$\mathrm{Ret}(012) = \{012,\ 012120,\ 012201,\ 012120201, 01212012, 0121202012, 012120201120201, 0120201, 01201\}$.
•$\mathrm{Ret}(0120) = \{0120121202011202010122010121 2, 01202010122010121200121202, 01201012120012120201120201, 012012120201120201012201012120201012120012120201120201, 0120201012201012120012120201012120201120201012201012 12, 01201012120012120201120201012120201012201012120012120 2, 012012120201120201012201012120012120201120201012201012 12, 012020101220101212001212020112020101220101212001212 02, 0120101212001212020112020101220101212001212020112020 1\}$.
•$\mathrm{Ret}(01201) = \{\mu_3(00v_{00}00)2^{-1}, \mu_3(00v_{01}11v_{12}22), \mu_3(00v_{01}11v_{12}22), 2^{-1}\mu_3(22v_{22}22), \mu_3(00v_{01}11u_{11}11v_{12}22), \mu_3(00v_{01}11v_{12}22), 2^{-1}\mu_3(22v_{20}00)2^{-1}, 2^{-1}\mu_3(22v_{20}00)2^{-1}\}$.

4. Privileged complexity of $\mathbf{t}_3$

In this section, we define a recursive function to calculate the privileged complexity of $\mathbf{t}_3$. This function is based on the privileged complexities of the first complete return words of 0.

Now, let us examine the first letters of $\mathbf{t}_3$ below:

$$\mathbf{t}_3 = \mu_3^\infty(0)$$
$$= \underbrace{012120}_{r_7} 2 \overbrace{01120}^{r_5} 2 \underbrace{010}_{r_2} 1220101212012 \underbrace{020}_{r_3}$$
$$1 \overbrace{01220}^{r_6} 101212 \underbrace{00}_{r_1} 121202 \overbrace{0120}^{r_4} 101212 \cdots$$

Thus, the set of the first complete return words of 0 in $\mathbf{t}_3$ is given by:

$$\mathcal{R} = \{r_1, r_2, r_3, r_4,\ r_5, r_6, r_7\}, \text{ where :}$$

$r_1 = 00, r_2 = 010, r_3 = 020, r_4 = 0120, r_5 = 01120, r_6 = 01220, r_7 = 012120.$

Lemma 5. *Let $w \in Pri_0(n)$ be such that $n \geq 7$. Then there exists an $u \in \mathcal{R}$ such that w begins with the first complete return word of u.*

Proof. Let $w \in Pri_0(n)$. Observe that $\mathcal{R}$ contains all privileged factors of length less than 7 starting with 0. Since $|w| \geq 7$, w admits a proper prefix $u \in \mathcal{R}$. Because w is privileged, it must also end with some $u \in \mathcal{R}$. Therefore, w contains at least two occurrences of u, implying that w begins with a first complete return word of u.

Theorem 2. *For any integer $n \geq 2$, the privileged complexity $c^{\mathrm{pri}}_{\mathbf{t}_3}(n)$ of $\mathbf{t}_3$ is given by:*

$$\frac{1}{3}c^{\mathrm{pri}}_{\mathbf{t}_3}(n) = c^{\mathrm{pri}}_{r_1}(n) + c^{\mathrm{pri}}_{r_2}(n) + c^{\mathrm{pri}}_{r_3}(n) +$$
$$c^{\mathrm{pri}}_{r_4}(n) + c^{\mathrm{pri}}_{r_5}(n) + c^{\mathrm{pri}}_{r_6}(n) + c^{\mathrm{pri}}_{r_7}(n).$$

Proof. Since $F_{\mathbf{t}_3}$ is stable under E_3, we consider factors starting with 0; the cases starting with 1 and 2 are treated similarly. The letters play symmetric roles, and the image of each letter begins with the letter itself and includes the other letters. Furthermore, because the substitution is 3-uniform, factors beginning with 1 (resp. 2) are obtained by applying E_3 (resp. E_3^2) to those beginning with 0. Thus, the privileged words of $\mathbf{t}_3$ starting with 0 and of length at least 2 are divided into seven groups according to the elements of $\mathcal{R}$. By Lemma 4, we have:

$$\mathrm{Pri}_0(n) = \bigcup_{k=1}^{7} \mathrm{Pri}_{r_k}(n).$$

The sets are disjoint because the elements of the sequence $(r_k)_{1\leq k\leq 7}$ are neither prefixes nor suffixes of each other. Therefore, the privileged complexity function of $\mathbf{t}_3$ satisfies:

$$\frac{1}{3}c^{\mathrm{pri}}_{\mathbf{t}_3}(n) = \sum_{k=1}^{7} c^{\mathrm{pri}}_{r_k}(n).$$

By considering the lengths of the privileged words of $\mathbf{t}_3$, we establish the following result:

Theorem 3. *The privileged complexity function* $c^{pri}_{\mathbf{t}_3}$ *of* $\mathbf{t}_3$ *verifies:*

$$c^{pri}_{\mathbf{t}_3}(n) = \begin{cases} 1 \text{ if } n = 0 \\ 3 \text{ if } n \in \{1,2,4,6\} \\ 6 \text{ if } n \in \{3,5\} \\ 3c^{pri}_{r_2}(n) + 3c^{pri}_{r_3}(n) + 3c^{pri}_{r_7}(n) \\ \quad \text{if } n \equiv 0[3], n \geq 9 \\ 3c^{pri}_{r_4}(n) \text{ if } n \equiv 1[3], n \geq 7 \\ 3c^{pri}_{r_1}(n) + 3c^{pri}_{r_5}(n) + 3c^{pri}_{r_4}(n) + \\ \quad 3c^{pri}_{r_5}(n) \text{ if } n \equiv 2[3], n \geq 8 \end{cases}$$

Proof. From $\mathcal{R}$, we have:

$$\mathrm{Pri}_0(1) = \{0\},\ \mathrm{Pri}_0(2) = \{r_1\},\ \mathrm{Pri}_0(3) = \{r_2, r_3\},$$
$$\mathrm{Pri}_0(4) = \{r_4\},\ \mathrm{Pri}_0(5) = \{r_5, r_6\},\ \mathrm{Pri}_0(6) = \{r_7\}.$$

Let w be a privileged word of $\mathbf{t}_3$ starting with 0 and of length $n \geq 7$. Since μ_3 is 3-uniform, the length of w can be expressed modulo 3. Moreover, by Lemma 4, the decomposition of w is unique because $|w| \geq 7$. According to Lemma 5, w begins and ends with an element of $\mathcal{R}$. Using the elements of $\mathcal{R}$, we consider the following cases.

Case 1. w begins and ends with $r_1 = 00$. Then w has the form $r_1 w' r_1$, where $w' \in F_{\mathbf{t}_3}$. The first 0 in r_1 is a suffix of $\mu_3(1)$, and the second 0 is a prefix of $\mu_3(0)$. By the synchronization lemma (Lemma 4), w can be written as $w = 0\mu_3(w'')0$, where $w'' \in F_{\mathbf{t}_3}$. Thus, $|w| = 3k+2$ since μ_3 is 3-uniform. Similarly, if w begins with $r_5 = 01120$ or $r_6 = 01220$, it synchronizes as $01\mu_3(w'')$ or $w''20$, respectively, with $w'' \in F_{\mathbf{t}_3}$. Therefore, when w begins with $r_1 = 00$, $r_5 = 01\mu_3(1)$, or $r_6 = \mu_3(0)20$, we have $|w| = 3k+2$ for some $k \geq 1$.

Case 2. w begins with $r_2 = 010$. Proceeding as in Case 1, we show that w synchronizes as $01\mu_3(w')0$ because 01 is the suffix of $\mu_3(2)$. Similarly, if w begins with $r_7 = 012120$, then w synchronizes as $01\mu_3(w')$ because $r_7 = \mu_3(01)$. By Theorem 1, any return word of $r_4 = 0120$ has length either $3k$ or $3k+2$. Hence, any first complete return word for r_4 has length either $3(k+1)+1$ or $3(k+1)+2$.

We now present some of the relationships between the privileged complexity functions of the elements in $\mathcal{R}$.

Proposition 2. *For all* $n \geq 9$, $c^{pri}_{r_1}(n) \leq c^{pri}_{r_7}(n-2)$.

Proof. Note that the word $r_1 = 00$ is always preceded and followed by 12120. The separator factors following r_1 are:

1212020, 1212020120101212, 1212020101212020.

Suppose $w \in \mathrm{Pri}_{r_1}(n)$. Then $0^{-1}w$ admits $r_7 = 012120$ as a privileged prefix. Similarly, $w0^{-1}$ has $r_7 = 012120$ as a privileged suffix. Therefore, $0^{-1}w0^{-1}$ has $r_7 = 012120$ as both a prefix and a suffix. It follows that

$$\mathrm{Pri}_{r_1}(n) \subset \mathrm{Pri}_{r_7}(n-2),$$

and consequently,

$$c^{pri}_{r_1}(n) \leq c^{pri}_{r_7}(n-2).$$

Proposition 3. *For all* $n \geq 8$, $c^{pri}_{r_5}(n) = c^{pri}_{r_1}(n-3)$.

Proof. Note that for $n \geq 8$, the words $r_1 = 00$, $E_3(r_1) = 11$, and $E_3^2(r_1) = 22$ have the same privileged complexity function. Since the square 11 in $\mathbf{t}_3$ is the border of the images $\mu_3(2)$ and $\mu_3(1)$ (specifically, $\mu_3(21) = 201120$), it is necessarily preceded by 0 and followed by 20. Therefore, any occurrence of $r_5 = 01120$ corresponds to an occurrence of 11. Thus, for any $w \in \mathrm{Pri}_{r_5}(n)$, we have $0^{-1}w(20)^{-1} \in \mathrm{Pri}_{11}(n-3)$. Consequently,

$$conc^{pri}_{r_5}(n) = c^{pri}_{r_1}(n-3).$$

Proposition 4. *For all* $n \geq 9$, $c^{pri}_{r_3}(n) = c^{pri}_{r_6}(3n-4)$.

Proof. Suppose $w \in \mathrm{Pri}_{r_3}(n)$. Then w can be written as $w = r_3 w' r_3$, where w' is a finite word occurring in $\mathbf{t}_3$.

Note that $r_6 = 01220$ is a proper prefix of $\mu_3(02) = \mathbf{012201}$. Furthermore, 02 is not a special factor of $\mathbf{t}_3$ because 022 is not a factor of $\mathbf{t}_3$. Therefore, any occurrence of

r_6 in $\mathbf{t}_3$ is a proper prefix of $\mu_3(r_3)$. As a result:

$$\begin{aligned}\mu_3(w) &= \mu_3(r_3 w' r_3)\\ &= \mu_3(r_3)\mu_3(w')\mu_3(r_3)\\ &= r_6\mathbf{1012}\mu_3(w')r_6\mathbf{1012}.\end{aligned}$$

Thus,

$$\mu_3(w)(\mathbf{1012})^{-1} \in \mathrm{Pri}_{r_6}(3n-4),$$

i.e.,

$$\mathrm{Pri}_{r_3}(n) \subset \mathrm{Pri}_{r_6}(3n-4).$$

Hence,

$$\mathsf{c}^{\mathrm{pri}}_{r_3}(n) \le \mathsf{c}^{\mathrm{pri}}_{r_6}(3n-4).$$

Conversely, suppose $v \in \mathrm{Pri}_{r_6}(3n-4)$. Then, by Lemma 4 and desubstitution of $\mu_3(v)$, it follows that $\mathrm{Pri}_{r_6}(3n-4) \supset \mathrm{Pri}_{r_3}(n)$. Therefore,

$$\mathsf{c}^{\mathrm{pri}}_{r_3}(n) \ge \mathsf{c}^{\mathrm{pri}}_{r_6}(3n-4).$$

Proposition 5. *For all $n \ge 9$, $\mathrm{c}^{\mathrm{pri}}_{r_4}(n) = \mathrm{c}^{\mathrm{pri}}_{ii}(3n-k)$, $k \in \{0,2,3\}$, $i \in \mathcal{A}_3$.*

Proof. We show that any factor of the form $i(i+1)(i+2)i$, where $i \in \mathcal{A}_3$, arises only from the image of a square of a letter under μ_3. These factors decompose as follows:

- $\mu_3(i)i = i(i+1)(i+2)i$, which is a prefix of $\mu_3(ii) = i(i+1)(i+2)i(i+1)(i+2)$.
- $i\mu_3(i+1) = i(i+1)(i+2)i$, which is a suffix of $\mu_3((i+1)(i+1)) = (i+1)(i+2)i(i+1)(i+2)i$.
- $[i(i+1)][(i+2)i]$, which is a median factor of $\mu_3((i+2)(i+2)) = (i+2)i(i+1)(i+2)i(i+1)$.

Let u be a privileged word starting with r_4, and let v_{ij} be the separator factor between the squares ii and jj. Depending on the values of i and j, we have the following cases:

Case 1. $i=0$: then $j=0$ or $j=1$.

- If $j=0$, then $u = \mu_3(00v_{00}00)(12)^{-1}$.
- If $j=1$, then $u = \mu_3(00v_{01}11)$.

Case 2. $i=1$: then $j=1$ or $j=2$.

- If $j=1$, then $u = (12)^{-1}\mu_3(11v_{11}11)$.
- If $j=2$, then $u = (12)^{-1}\mu_3(11v_{12}22)(1)^{-1}$.

Case 3. $i=2$: then $j=2$ or $j=0$.

- If $j=2$, then $u = (2)^{-1}\mu_3(22v_{22}22)(1)^{-1}$.
- If $j=0$, then $u = (2)^{-1}\mu_3(22v_{20}00)(12)^{-1}$.

Consequently, $u = \delta_1^{-1}\mu_3(iiv_{ij}jj)\delta_2^{-1}$, where $|\delta_1\delta_2| \in \{0,2,3\}$. Setting $k = |\delta_1\delta_2|$, we obtain

$$\mathsf{c}^{\mathrm{pri}}_{r_4}(n) = \mathsf{c}^{\mathrm{pri}}_{ii}(3n-k).$$

5. Conclusion

This paper aimed to generalize results on privileged words from $\mathbf{t}_2$ to $\mathbf{t}_3$. To achieve this, we constructed a set $\mathcal{R}$ and derived a complete recursive formula for calculating the privileged complexity of $\mathbf{t}_3$. A natural direction for future work is to generalize our recursive formula to the generalized Thue-Morse word $\mathbf{t}_{b,q}$, generated over the alphabet $\mathcal{A}_q = \{0,1,\ldots,q-1\}$ by the morphism $\mu_{p,q}$ defined as:

$$\mu_{p,q}(k) = k(k+1)\ldots(k+p-1),$$

where $b \ge 2$, $q \ge 3$, and letters are taken modulo q. We plan to investigate the relationship between the recursive functions and the substitution parameters b and q.

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