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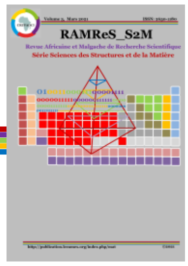

Numerical analysis of an asymptotic model for Kazhikhov-Smagulov type equations

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Abstract:

In this paper, we develop a hybrid scheme that combines finite volume and finite element methods for an asymptotic model of the Kazhikhov-Smagulov equations. We first establish the stability of the proposed scheme and its convergence toward the weak solution of the problem. Numerical simulations are then performed to verify the scheme's robustness and confirm the convergence of the solution.

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1. Introduction

The equations considered in this study are those of the Kazhikhov-Smagulov model, which describes the behavior of a homogeneous mixture of two viscous and incompressible fluids in a subset of $\mathbb{R}^3$ over a time interval $]0\ ;\ T[$, under a mass effect following Fick's law. Let W denote the barycentric speed or the average mass speed of the mixture; the conservation laws for mass and momentum lead to the following differential equations defined in $Q_T = [0\ ;\ T[\times \Omega$ with $T > 0$ and Ω an open subset of $\mathbb{R}^3$ (see [1]).

$$\begin{cases} \partial_t \rho + \mathrm{div}\ \rho W \ = 0, \\ \rho\ \partial_t W + W.\nabla W \ - \mu \Delta W + \nabla p \\ \quad - \ \mu + \mu'\ \nabla \mathrm{div}\ W \ = \rho f, \end{cases} \quad (1)$$

where p represents the pressure, $f = -g\ \vec{k}$ with g the gravitational acceleration, $\vec{k} = {}^t(0, 0, 1)$, μ and μ' are assumed to be constants with $\mu > 0$ and $2\mu + 3\mu' \geq 0$, ρ is the density of the mixture, and μ characterizes the dynamic viscosity.

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Let us denote by $V = (u, v, w)$ the average volume velocity of the mixture, where $U = (u, v)$ represents the horizontal components of V. In this framework, Fick's law is expressed as

$$W = V - \frac{\lambda}{\rho}\nabla\rho \tag{2}$$

where λ is the mass diffusion coefficient. Using this relation, one can derive the following system:

$$\begin{cases} \partial_t\rho + V\cdot\nabla\rho = \lambda\Delta\rho \\ \rho\Big(\partial_t V + (V\cdot\nabla)V\Big) - \mu\Delta V - \lambda(\nabla\rho\cdot\nabla)V \\ -\lambda(V\cdot\nabla)\nabla\rho + \lambda^2\Big(\nabla\rho\cdot\nabla\Big(\frac{\nabla\rho}{\rho}\Big) - \frac{\Delta\rho}{\rho}\nabla\rho\Big) \\ +\nabla P = \rho f \\ \mathrm{div}(V) = 0 \end{cases}$$

Following the asymptotic analysis performed in [2], this system can be rewritten in a reduced form involving only $U = (u, v)$:

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho U) = \dfrac{c}{Re}\Delta\rho \\ \rho\Big(\partial_t U + (U\cdot\nabla)U\Big) - \dfrac{1}{Re}\Delta U - \dfrac{c}{Re}\Big((\nabla\rho\cdot\nabla)U \\ +(U\cdot\nabla)\nabla\rho\Big) + \dfrac{1}{Fr^2}\rho\nabla\rho = 0 \\ \mathrm{div}(U) = 0 \end{cases} \tag{3}$$

where ρ denotes the density, $U = (u, v)$ is the two-dimensional velocity vector, $c = \dfrac{\lambda\bar{\rho}}{\mu}$ with λ the mass diffusion coefficient, $\bar{\rho}$ a reference density, and μ the dynamic viscosity of the mixture. Finally, Re is the Reynolds number and Fr the Froude number.

The incompressibility condition is expressed as $\mathrm{div}(U) = 0$.

To the system (3), we add the following boundary and initial conditions:

$$\begin{cases} \dfrac{\partial\rho}{\partial n} = 0 \quad \text{on} \quad \Sigma_T \\ U = 0 \quad \text{on} \quad \Sigma_T \\ \rho_{|t=0} = \rho_0\,;\; U_{|t=0} = U_0 \quad \text{in} \quad Q, \end{cases} \tag{4}$$

where $\Sigma_T = [0, T]\times\partial Q$, and Q is an open subset of $\mathbb{R}^2$.

Few studies address the numerical simulation of non-homogeneous Navier-Stokes equations, also known as Kazhikhov-Smagulov equations [1]. However, existing works [3–7] provide useful references. Most approaches rely on finite volume or finite element methods, ensuring stability and convergence.

For instance, [4] analyzes an Euler-FEM scheme for mass diffusion, while [6, 7] establish stability for discrete models with density truncation. [7] also studies asymptotic behavior as $\lambda \to 0$, leading to a weak solution. The full model with $\mathcal{O}(\lambda^2)$ terms is explored in [3]. Hybrid schemes, combining finite volume and finite element methods, were introduced by Catarina Calgaro *et al.* [8] for incompressible flows and later extended to other models [9, 10].

This work follows two main directions. First, a hybrid scheme is developed to discretize (3)-(4), using finite volume for mass conservation and finite elements for momentum. The mass equation employs a two-step finite volume method with a second-order Runge-Kutta scheme, while velocity evolution follows an Euler scheme.

Let $h > 0$, we denote by τ_h a partition of Q composed of conforming and isotropic triangles. We take $\mathcal{W}_h\times\mathcal{V}_h \subset H^1(Q)\times H^1_0(Q)$ the finite elements spaces associated with density and velocity, respectively. For the simplification of notation, we restrict our study to the case of uniform time discretization of $[0, T]$. Let N be a positive integer, then we define $\Delta t = T/N$ the time step and $(t^n = n\Delta t)_{n=0}^N$ the partition of $[0, T]$. Moreover, we consider the following stability condition:

$$0 < \Delta t \le c_0 h \tag{5}$$

where $c_0 > 0$ is a constant that is independent of h and Δt, but depends on the velocity field $U \in \mathcal{V}_h$. Clearly, (5) is a typical CFL condition often used for the numerical solution of conservation laws (see [11]). Let $(\rho_h^n, U_h^n) \in \mathcal{W}_h \times \mathcal{V}_h$ be the approximation of density and velocity at time t^n. We denote by $\rho_{h,\Delta t}$ and $U_{h,\Delta t}$ the piecewise constant functions in time taking values ρ_h^n and U_h^n on $(t^{n-1}, t^n]$, respectively.

The second direction of this work aims to demonstrate the existence of a unique solution $(\tilde{\rho}_h^{n+1}, \rho_h^{n+1}, U_h^{n+1})$ for the scheme and establish its convergence to a weak solution (ρ, U) of the system (3)-(4).

Thus, this paper is structured as follows: the first section introduces the model and key existing results; the second details the hybrid scheme construction using finite volume and finite element methods; the third establishes its properties, including stability, a priori estimates, and convergence analysis and the last one is dedicated to numerical results.

2. Preliminary

2.1. Notations

In this section, we define specific functional spaces for the Kazhikhov-Smagulov model and introduce the concept of weak solution for the asymptotic model. Set

$$H = \left\{U \in L^2(Q)^2 : \quad U.n = 0 \quad \text{on} \quad \partial Q\right\}$$

and

$$V = \left\{U \in H^1(Q)^2 : \quad U = 0 \quad \text{on} \quad \partial Q\right\}.$$

On the other hand, we consider the analogous space,

$$H_N^2(Q) = \left\{\rho \in H^2(Q) : \frac{\partial \rho}{\partial n} = 0 \quad \text{on} \; \partial Q, \int_Q \rho(x) = \int_Q \rho_0\right\},$$

with $H_N^2 = \bar{\rho} + H_{N,0}^2(Q)$
where $\bar{\rho} = \frac{1}{mes(Q)} \int_Q \rho_0$ and

$$H_{N,0}^2 = \left\{\rho \in H^2(Q) : \frac{\partial \rho}{\partial n} = 0 \quad \text{on} \; \partial Q, \int_Q \rho(x) = 0\right\}.$$

It is shown that $H_{N,0}^2$ is a closed subspace of H^2, the norms $\|\rho\|_{H^2(Q)}$ and $\|\Delta\rho\|_{L^2(Q)}$ are equivalent (see for instance [2], [12] and [13] for their properties). Throughout this work, the scalar product will be denoted by $\langle \cdot, \cdot \rangle$.

2.2. Setting of the problems

We consider initial conditions u_0 and ρ_0 such that:

$$u_0 \in H, \qquad \rho_0 \in H^1(Q) \cap L^\infty(Q) \tag{6}$$

$$0 < m \leq \rho_0 \leq M < +\infty \tag{7}$$

Let us recall the definition of the weak solution of the system (3).

Definition 2.1. *Let $\rho_0 \in H^1(Q) \cap L^\infty(Q)$, $U_0 \in H$. A pair (U, ρ) is said to be a solution of the asymptotic model (3)-(4) in Q if:*

$$\rho \in L^2(0,T; H_N^2(Q)) \cap L^\infty(Q_T) \cap L^\infty(0,T; H_N^1(Q)) \tag{8}$$

$$U \in L^2(0,T;V) \quad ; \; \rho U \in L^\infty(0,T; L^2(Q)^2) \tag{9}$$

and satisfies, $\forall \psi \in \mathcal{D}(Q_T)$

$$-\int_{Q_T} \left[\rho \partial_t \psi + (\rho U - \frac{c}{Re} \nabla \rho).\nabla \psi\right] = \int_Q \rho_0 \psi(0,x,y)dxdy \tag{10}$$

$$-\int_{Q_T} \Big[\rho U \partial_t \psi + (\rho U \otimes U - \frac{1}{Re} \nabla U - \frac{c}{Re}(\nabla \rho \otimes U + U \otimes \nabla \rho)).\nabla \psi\Big] - \frac{1}{2Fr^2} \int_{Q_T} \rho^2 \operatorname{div}(\psi) + \frac{c}{Re} \int_{Q_T} \operatorname{div}(U) \nabla \rho.\psi = \int_Q \rho_0 U_0 \psi(0,x,y)dxdy \tag{11}$$

The following lemma guarantees the boundedness of the density ρ.

Lemma 2.1. *[2](Maximum Principle)*
Let ρ define in $]0,T[\times Q$, such that:
$\partial_t\rho + \text{div}(\rho U) = \lambda\Delta\rho$ *with* $\text{div}(U) = 0$*;*
$\rho(0) = \rho_0$, $0 < m \le \rho_0 \le M < +\infty$.
Then

$$0 < m \le \rho(x,y,t) \le M < +\infty.$$

Now, we give the existence result of the weak solution to the model (3).

Theorem 2.1. *[2] Let $u_0 \in H$; $\rho_0 \in H^1(Q)\cap L^\infty(Q)$.*

Suppose that $\rho_0 \ge 0$ almost everywhere in Q and $0 \le \lambda \le \dfrac{\mu}{4M}$.

Then, there exists at least a weak solution (ρ, U) of the asymptotic dispersion mass model (3) in Q_T.

Next, we derive the variational formulation. Assuming (ρ, U) is a regular solution of (3)-(4), we multiply the equations by test functions $(\xi,\vartheta,\pi) \in H^1(Q)\times H_0^1(Q)\times H_0^1(Q)$, integrate over Q, and apply Green's theorem. Adding the density equation to momentum with $\xi = \frac{1}{2}u.\vartheta$ and integrating by parts the convective and diffusive terms yield the following formulation for a.e. $t \in (0;T)$: Find $(\rho, U, \mathcal{P}) \in H^1(Q)\times V\times H_0^1(Q)$ such that $\forall(\xi,\vartheta,\pi) \in H^1(Q)\times V\times H_0^1(Q)$, we have

$$\begin{cases} \langle\partial_t\rho,\xi\rangle + b(\rho,\xi,u) - \dfrac{c}{Re}\langle\nabla\rho,\nabla\xi\rangle = 0, \\ \forall\xi \in H^1(Q) \\ \langle\rho\partial_t U,\vartheta\rangle + \dfrac{1}{2}\langle U\partial_t\rho,\vartheta\rangle + a(\rho,U,\vartheta) \\ \quad +c(\rho U - \dfrac{c}{Re}\nabla\rho, U,\vartheta) - \langle\mathcal{P},\nabla\vartheta\rangle = 0, \\ \forall\vartheta \in H_0^1(Q) \\ d(U,\pi) = 0, \quad \forall\pi \in H_0^1(Q) \\ U = 0, \quad \dfrac{\partial\rho}{\partial n} = 0, \quad \text{on} \quad \Sigma_T \end{cases} \tag{12}$$

with, $\mathcal{P} = -\dfrac{c}{Re}(U.\nabla\rho) + \dfrac{1}{2Fr^2}\rho^2$.
Here we have used the following identity ([4]) in the momentum equation

$$-\frac{c}{Re}(U.\nabla)\nabla\rho = -\frac{c}{Re}\nabla(U.\nabla\rho) + \frac{c}{Re}\nabla.(\rho(\nabla U)^t),$$

and the following notations:

- $b(.,.,.)$; $a(.,.,.)$ and $c(.,.,.)$ are the trilinear forms defined by:

$$b(\rho,\beta,U) = \int_Q \text{div}(\rho U)\beta dx,$$
$$\forall\rho \in H^1(Q)\cap L^\infty(Q),\ \beta \in H^1(Q),\ U \in V \tag{13}$$

$$a(\rho,U,\vartheta) = \frac{1}{Re}(\nabla U,\nabla\vartheta) - \frac{c}{Re}\int_Q \Big(\rho - \frac{\tilde{M}+\tilde{m}}{2}\Big)(\nabla U)^t : \nabla\vartheta dx,$$
$$\forall\rho \in H^1(Q) \tag{14}$$
$$\forall U,\vartheta \in H_0^1(Q),$$

$$c(\omega,U,\vartheta) = \frac{1}{2}\Big[((\omega.\nabla)U,\vartheta) - ((\omega.\nabla)\vartheta,U)\Big],$$
$$\forall\omega \in V,\ \ U,\vartheta \in H_0^1(Q) \tag{15}$$

- $d(.,.)$ is a bilinear form defined by:
$$d(U,\pi) = -(p,\text{div}(U)),\ \forall\, U \in V,\ p,\ \pi \in H_0^1(Q) \tag{16}$$

The trilinear forms verify the following properties of continuity, coercivity and antisymmetry as in [3,6,7,10]:
There exists $\alpha > 0$ and $C > 0$ such that:

$$a(\rho,U,U) \ge \alpha\|\nabla U\|^2_{L^2(Q)},\ \forall U \in H_0^1(Q) \tag{17}$$
$$a(\rho,U,v) \le C\|U\|_{H_0^1(Q)}\|v\|_{H_0^1(Q)},\ \forall U,v \in H_0^1(Q) \tag{18}$$
$$c(\omega,U,v) \le C\|\omega\|_{L^3(Q)}\|u\|_{H_0^1(Q)}\|v\|_{H_0^1(Q)},\ \forall\omega \in L^3(Q),\forall U,v \in H_0^1(Q) \tag{19}$$
$$c(\omega,U,v) = -c(\omega,v,U),\ \forall\omega \in V,\ \forall U,v \in H_0^1(Q) \tag{20}$$
$$c(\omega,U,U) = 0,\ \forall\omega \in V,\forall U \in H_0^1(Q) \tag{21}$$

3. The hybrid finite volume-element method

The hybrid scheme uses a time-splitting approach: finite volume for mass conservation and finite element for momentum under incompressibility. It builds on prior work [8–10,14] for similar Navier-Stokes models.

3.1. The time splitting

Let Δt be the time step and $t^n = n\Delta t$. The approximate variables at time t^n are identified with the exponent n. Suppose that ρ^1 and U^1 have been computed by an Euler scheme. Assume also that for $n \geq 1$, ρ^{n-1} and U^{n-1} as well as ρ^n and U^n are known. Now, let's compute ρ^{n+1} and U^{n+1}.

1. We begin by evaluating ρ^{n+1} by solving the mass conservation equation using a second-order Runge-Kutta scheme:

$$\frac{\tilde{\rho}^{n+1} - \rho^n}{\Delta t} + \nabla \cdot (\rho^n U^{n+\frac{1}{2}}) = \frac{c}{Re}\Delta\tilde{\rho}^{n+1} \quad (22)$$

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} + \frac{1}{2}\Big(\nabla \cdot (\rho^n U^{n+\frac{1}{2}}) + \nabla \cdot (\tilde{\rho}^{n+1} U^{n+\frac{1}{2}})\Big) = \frac{c}{Re}\Delta\rho^{n+1} \quad (23)$$

with

$$U^{n+\frac{1}{2}} = \frac{3U^n - U^{n-1}}{2} \quad (24)$$

And at the boundary

$$\tilde{\rho}^{n+1}(x) = \rho^{n+1}(x) = \rho^{n+1}_{|\sum_T}, \quad \forall x \in \partial Q.$$

These two relations, (22) and (23), result from the application of a second-order Runge-Kutta scheme of the predictor-corrector type (Heun's method) to the advection-diffusion equation for the density, with the velocity field frozen at the half-step $U^{*,n+\frac{1}{2}}$. The first step (22) corresponds to a prediction obtained using an explicit Euler method for the convection term and an implicit method for the diffusion term, whereas the second step (23) performs the correction by employing the trapezoidal average of the convective fluxes, with the diffusion term remaining implicit.

2. We then calculate U^{n+1} by solving the momentum conservation equation and the incompressibility constraint of the mixture using an Euler scheme.

$$\begin{cases} \rho^n \left(\dfrac{U^{n+1} - U^n}{\Delta t} + \Big((2U^n - U^{n-1}).\nabla\Big) U^{n+1}\right) \\ -\dfrac{c}{Re}\Big(\Big(\nabla\rho^{n+1}.\nabla\Big) U^{n+1} + \Big((2U^n - U^{n-1}).\nabla\Big) \\ \nabla\rho^{n+1}\Big) - \dfrac{1}{Re}\Delta U^{n+1} + \dfrac{1}{Fr^2}\rho^{n+1}\nabla\rho^{n+1} = 0, \\ \nabla \cdot U^{n+1} = 0, \\ U^{n+1}|_{\sum_T} = 0. \end{cases} \quad (25)$$

In this relation, the density appearing in the term multiplying the time derivative of the velocity is taken at time step n, while U^{n+1} is computed using an implicit Euler scheme. For stability reasons, however, in the nonlinear terms where U^{n+1} appears, it is approximated by the extrapolation formula $U^{n+1} = 2U^n - U^{n-1}$.

Remark 3.1. • *Following [9], it should be noted that the velocity $U^{n+\frac{1}{2}}$ in equation (24) is an extrapolation of the velocity at time $(t^{n+1} - t^n)/2$, which is necessary to achieve second-order accuracy in the Runge-Kutta scheme.*

- *The quantity* $(2U^n - U^{n-1})$ *in equation* (25) *is an explicit Euler scheme chosen to approximate the velocity at time* t^{n+1}.

3.2. The spatial discretization

1. **Construction of the mesh**

   Let $Q \subset \mathbb{R}^2$ be a bounded polygonal domain with boundary ∂Q, and $[0, T]$ the time interval. The mass conservation equation is discretized on an unstructured triangular mesh τ_h of Q, composed of L conforming isotropic triangles, where L depends on the mesh refinement.

   We assume the following hypotheses:

   (H1) Let $\{\tau_h\}_{h>0}$ be a regular family of triangulations of Q.

   (H2) The triangulation τ_h is weakly acute, meaning no triangle has an angle exceeding $\pi/2$.

   We denote h as the spatial step of the triangulation defined by:

$$h = \max_{K \in \tau_h}(h(K)) \qquad (26)$$

   where K represents an arbitrary triangle of the triangulation τ_h, and $h(K)$ is the length of the longest side of the triangle $K \in \tau_h$, as defined in [10].

   (H3) The triangulation τ_h satisfies the following inverse hypothesis:

$$h \leq ch(K), \qquad \forall K \in \tau_h,$$

   where $c > 0$ is a constant independent of h.

   According to the reference [15], assumptions **(H1)** and **(H3)** lead to the existence of a constant $c > 0$ independent of h such that,

$$h^2 \leq c|K|, \qquad \forall K \in \tau_h \qquad (27)$$

   where $|K|$ represents the area of $K \in \tau_h$. For each element $K \in \tau_h$, B_K is the barycenter, and M_i, M_{j_1}, M_{j_2} are the triangle vertices. M_{ij_1} and M_{ij_2} are the midpoints of $[M_i M_{j_1}]$ and $[M_i M_{j_2}]$, respectively. The dual mesh $C_h = \{\mathcal{C}_i, i \in [1, I]\}$ partitions Q, where I is the number of vertices of $K \in \tau_h$. The dual finite volume $\mathcal{C}_i$ for each vertex M_i is a polygon formed by connecting B_K to the midpoints of the sides of K sharing M_i, and completed with boundary segments if $M_i \in \partial Q$. $\mathcal{C}_i$ is the control volume around M_i. As a consequence, we have:

$$\bigcup_{K \in \tau_h} = \bar{Q} = \bigcup_{i \in J} \mathcal{C}_i.$$

   Moreover, we have

$$|\mathcal{C}_i| = \sum_{K, M_i \in K} \frac{|K|}{3} \qquad (28)$$

   For $i \in J$, let $\mathcal{V}(i) = \{j \in J, \mathcal{C}_j$ is a neighbor of $\mathcal{C}_i\}$. Let $i \in J$ and $j \in \mathcal{V}(i)$, we define $K_{ij,1}$ and $K_{ij,2}$ as two neighboring triangles of τ_h sharing the same edge. We denote B_1 (resp. B_2) as the barycenter of $K_{ij,1}$ (resp. $K_{ij,2}$) and M_{ij} as the midpoint of $[M_i M_j]$.

   Then, we define

$$\Gamma_{ij,1} = [M_{ij} B_1] \quad \text{and} \quad \Gamma_{ij,2} = [M_{ij} B_2].$$

   We also denote $n_{ij,1}$ (resp. $n_{ij,2}$) as the outward normal of $\mathcal{C}_i$ to $\Gamma_{ij,1}$ (resp. $\Gamma_{ij,2}$) and $|\Gamma_{ij,1}|$ (resp. $|\Gamma_{ij,2}|$) as the length of the segment $\Gamma_{ij,1}$ (resp. $\Gamma_{ij,2}$). For any $\mathcal{C}_i \in \mathcal{C}_h$, its boundary is given by

$$\partial \mathcal{C}_i = \bigcup_{j \in \mathcal{V}(i)} \Big(\Gamma_{ij,1} \cup \Gamma_{ij,2} \Big) \qquad (29)$$

   Thus, we have,

$$|\Gamma_{ij,l}| \leq \frac{h}{2} \quad \text{for } l = 1, 2. \qquad (30)$$

Consequently, there exists a constant $c_1 > 0$ such that:

$$|\partial \mathcal{C}_i| \leq c_1 h, \qquad \forall i \in J \qquad (31)$$

Therefore, (5), (30), and (31) imply the existence of a constant $c_2 > 0$, such that;

$$\frac{|\mathcal{C}_i|}{|\partial \mathcal{C}_i|} \geq c_2 h, \qquad \forall i \in J \qquad (32)$$

2. **Construction of discrete spaces**
The spatial discretization uses a triangulation of $Q \subset \mathbb{R}^2$ by a regular mesh τ_h. The velocity U_h is discretized with $\mathbb{P}_2$-Lagrange elements, and the density ρ_h with piecewise constants on the dual mesh τ_h^*. This dual mesh allows for a vertex-based finite volume scheme for mass conservation. The density field can also be viewed as a $\mathbb{P}_1$-Lagrange finite element field, with a value at each triangle node.

$$\mathcal{V}_h = \Big\{u_h \in \mathcal{C}^0(\bar{Q}_h)\backslash : v_{h|K} \in \mathcal{Q}(K), \quad \forall K \in \tau_h\Big\} \cap H_0^1(Q),$$

$$\mathcal{W}_h = \Big\{\beta_h \in \mathcal{C}^0(\bar{Q}_h)\backslash : \ \beta_{h|K} \in \mathcal{P}(K), \quad \forall K \in \tau_h\Big\} \cap H^1(Q).$$

Here, the spaces $\mathcal{Q}(K)$ and $\mathcal{P}(K)$ are polynomial spaces of degree p and q, respectively. Therefore, for our simulations, $\mathcal{Q}(K) = \mathbb{P}_2$ and $\mathcal{P}(K) = \mathbb{P}_1$.
Troughout this work, we suppose the following hypotheses [3,6,7,10,15]:

(H4) Regularity for the data:
We suppose that $U_0 \in \mathcal{V}_h$, $\rho_0 \in H^1(Q)$ with $0 < m \leq \rho_0 \leq M < +\infty$ in Q.

(H5) The triangulation τ_h of Q and the finite elements space $\mathcal{W}_h$ verify the following inverse inequality:

$$\|\nabla \xi_h\|_{L^2(Q)} \leq C h^{-1} \|\xi_h\|_{L^2(Q)}, \forall \xi_h \in \mathcal{W}_h \qquad (33)$$

(H6) Inf-sup-condition:
There exists a constant $C > 0$ independent of h, such that:

$$\inf_{p_h \in \mathcal{W}_h} \sup_{v_h \in \mathcal{V}_h - \{0\}} \frac{d(v_h, p_h)}{\|p_h\|_{L^2(Q)} \|\nabla v_h\|_{L^2(Q)}} \geq C.$$

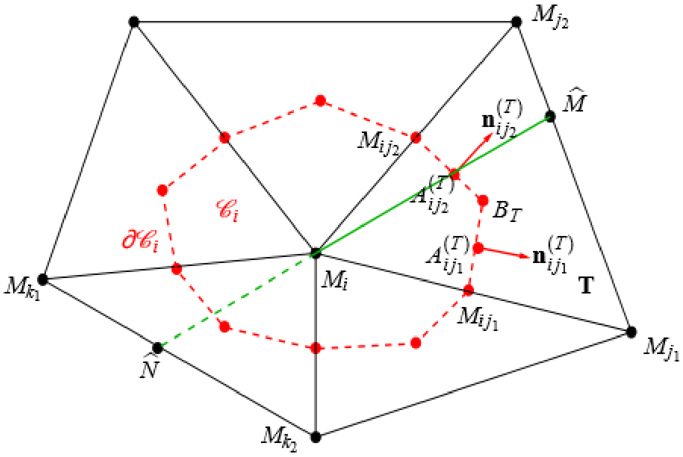

Fig. 1. Meshing the Q domain into triangles (See [14]).

The following proposition establishes a result of equivalence between the norms $\|.\|_h$ and $\|.\|_{L^2(\Omega)}$, which is essential for deriving the a priori estimates.

Proposition 3.1. *[10] There exists constants* $\hat{c}_1,\ \hat{c}_2 > 0$ *such that* $\forall h \in (0, h_0)$,

$$\hat{c}_1\|\beta\|_{L^2(Q)} \le \|\beta\|_h \le \hat{c}_2\|\beta\|_{L^2(Q)},\ \forall \beta \in \mathcal{W}_h.$$

3.3. The finite volume scheme

Using the density flux determination procedure for the finite volume scheme in [10], we obtain the finite volume scheme for the mass conservation equation using the second-order Runge-Kutta scheme as the temporal derivative scheme. Thus, we have:

$$\frac{\tilde{\rho}_h^{n+1} - \rho_h^n}{\Delta t} + \nabla \cdot \left(\rho_h^n U_h^{*,n+\frac{1}{2}}\right) = \frac{c}{Re}\Delta\tilde{\rho}_h^{n+1} \tag{34}$$

$$\frac{\rho_h^{n+1} - \rho_h^n}{\Delta t} + \frac{1}{2}\left(\nabla \cdot \left(\rho_h^n U_h^{*,n+\frac{1}{2}}\right) + \nabla \cdot \left(\tilde{\rho}_h^{n+1} U_h^{*,n+\frac{1}{2}}\right)\right) = \frac{c}{Re}\Delta\rho_h^{n+1} \tag{35}$$

where

$$U_h^{*,n+\frac{1}{2}} = \frac{3U_h^{*,n} - U_h^{*,n-1}}{2} \tag{36}$$

$$U_h^{*,n} = \frac{1}{|C_h|}\int_{C_h} U(t^n, x)dx \tag{37}$$

and

$$\rho_h^n = \frac{1}{|C_h|}\int_{C_h} \rho(t^n, x)dx \tag{38}$$

With the application of the flux determination method as in [10], this gives us:

$$\left\langle \frac{\tilde{\rho}_h^{n+1} - \rho_h^n}{\Delta t}, \xi_h \right\rangle + b_h\left(\rho_h^n, \xi_h, U_h^{*,n+\frac{1}{2}}\right) + \frac{c}{Re}\left\langle \nabla\tilde{\rho}_h^{n+1}, \nabla\xi_h \right\rangle = 0 \tag{39}$$

$$\left\langle \frac{\rho_h^{n+1} - \rho_h^n}{\Delta t}, \xi_h \right\rangle + \frac{1}{2}\left(b_h\left(\rho_h^n, \xi_h, U_h^{*,n+\frac{1}{2}}\right) + b_h\left(\tilde{\rho}_h^{n+1}, \xi_h, U_h^{*,n+\frac{1}{2}}\right)\right) + \frac{c}{Re}\left\langle \nabla\rho_h^{n+1}, \nabla\xi_h \right\rangle = 0 \tag{40}$$

where the trilinear form b_h is defined by (13). In conclusion, we define the finite volume scheme for the approximation of the solution $(\tilde{\rho}_h^{n+1}, \rho_h^{n+1})$ for equation (1) of system (3) as follows:

Initialization: Let $\rho_h^0 \in \mathcal{W}_h$ be the approximation of the initial solution ρ_0, with:

$$\rho_{M_i}^0 = \frac{1}{|C_i|}\int_{C_i} \rho_0(x)dx \tag{41}$$

At time step n+1: Let $\rho_h^n \in \mathcal{W}_h$ and $U_h^{*,n+\frac{1}{2}} \in \mathcal{V}_h$, find $(\tilde{\rho}_h^{n+1}, \rho_h^{n+1}) \in \mathcal{W}_h \times \mathcal{W}_h$, such that for all $\xi_h \in \mathcal{W}_h$, we have:

$$\begin{cases} \left\langle \frac{\tilde{\rho}_h^{n+1} - \rho_h^n}{\Delta t}, \xi_h \right\rangle + b_h\left(\rho_h^n, \xi_h, U_h^{*,n+\frac{1}{2}}\right) \\ +\frac{c}{Re}\left\langle \nabla\tilde{\rho}_h^{n+1}, \nabla\xi_h \right\rangle = 0 \\ \\ \left\langle \frac{\rho_h^{n+1} - \rho_h^n}{\Delta t}, \xi_h \right\rangle + \frac{1}{2}\left(b_h\left(\rho_h^n, \xi_h, U_h^{*,n+\frac{1}{2}}\right)\right. \\ \left.+b_h\left(\tilde{\rho}_h^{n+1}, \xi_h, U_h^{*,n+\frac{1}{2}}\right)\right) + \frac{c}{Re}\left\langle \nabla\rho_h^{n+1}, \nabla\xi_h \right\rangle \\ = 0 \end{cases} \tag{42}$$

Now, we examine specific characteristics of the density in the finite volume scheme described by equations (42). Since the finite volume scheme is linear and considering the properties of the trilinear form b_h, the existence and uniqueness of the solution $(\tilde{\rho}_h^{n+1}, \rho_h^{n+1})$ are guaranteed by the Lax-Milgram theorem. To achieve this, the following results are useful.

Lemma 3.1. *[7] There exists a constant* $C = C(Q) > 0$ *(independent of h) such that* *for all $\rho_h \in \mathcal{W}_h$, we have:*

$$\|\nabla \rho_h\|_{L^4(Q)} \leq C\|\nabla \rho_h\|^{\frac{1}{2}}_{L^2(Q)}\|\Delta_h \rho_h\|^{\frac{1}{2}}_{L^2(Q)}.$$

This lemma is a form of the Gagliardo-Nirenberg inequality used to augment the density gradient. We can formulate in the same way an a priori estimates for the velocity.

Lemma 3.2. *[10] There exists a constant $C > 0$ such that for $\rho, \beta \in \mathcal{W}_h$ and $\mathcal{V}_h$, we have:*

$$b_h(\rho, \beta, U)| \leq C\|U\|_{L^4(Q)}\|\nabla \rho\|_{L^4(Q)}\|\beta\|_{L^2(Q)}.$$

Furthermore, we have the following proposition:

Proposition 3.2. *The two equations of (42) are equivalent to the following equations:*

$$\tilde{\rho}^{n+1}_{M_i} + \frac{c}{Re}\frac{\Delta t}{|C_i|}\sum_{j\in\mathcal{V}(i)} a_{ij}\tilde{\rho}^{n+1}_{M_j} = \rho^n_{M_i} - \frac{\Delta t}{|C_i|} \left(\sum_{j\in\mathcal{V}_h(i)}\sum_{l=1}^{2}|\Gamma_{ijl}|G_{ijl}(\rho^n_{M_i}, \rho^n_{M_j}, n_{ijl})\right) \quad (43)$$

$$\rho^{n+1}_{M_i} + \frac{c}{Re}\frac{\Delta t}{|C_i|}\sum_{j\in\mathcal{V}(i)} a_{ij}\rho^{n+1}_{M_j} = \rho^n_{M_i} - \frac{\Delta t}{2|C_i|} \left(\sum_{j\in\mathcal{V}(i)}\sum_{l=1}^{2}|\Gamma_{ijl}|G_{ijl}(\tilde{\rho}^{n+1}_{M_i}, \rho^n_{M_j}, n_{ijl}) + \sum_{j\in\mathcal{V}(i)}\sum_{l=1}^{2}|\Gamma_{ijl}|G_{ijl}(\rho^n_{M_i}, \rho^n_{M_j}, n_{ijl})\right) \quad (44)$$

Proof. The proof follows the same lines as in [10] to obtain the finite volume scheme with the Euler scheme for the temporal derivative.

To obtain our results, it is crucial to guarantee that the preceding finite volume scheme maintains the maximum principle. The proposition below asserts the L^∞ stability of equations (39)-(40) on an unstructured grid, provided certain angle conditions and under an appropriate CFL condition. These stability properties, together with the a priori estimates established earlier, play a fundamental role in deriving both weak and strong convergence of the sequence of approximate solutions.

Proposition 3.3. *Let $U \in \mathcal{V}_h$ be the velocity satisfying the incompressibility condition, and let ρ_0 be the initial density satisfying the maximum principle. If the condition:*

$$0 < \Delta t < c_3 \frac{|C_i|}{|\partial C_i|}, \quad (45)$$

are satisfy, where $c_3 > 0$ is a constant. For all $0 < n < N - 1$, there exists a unique discrete solution $(\tilde{\rho}^{n+1}_h, \rho^{n+1}_h)$ for the finite volume scheme (39)-(40) that satisfies the pointwise estimates:

$$0 < m \leq \rho^{n+1}_h \leq M < +\infty$$
$$\text{and} \quad 0 < m \leq \tilde{\rho}^{n+1}_h \leq M < +\infty.$$

Proof. We can rewrite equations (42) follows:

$$\begin{cases} a_1\left(\tilde{\rho}^{n+1}_h, \xi_h\right) = l_1(\xi_h), \\ a_2\left(\rho^{n+1}_h, \xi_h\right) = l_2(\xi_h), \\ a_1\left(\tilde{\rho}^{n+1}_h, \xi_h\right) = \left\langle \tilde{\rho}^{n+1}_h, \xi_h\right\rangle + \frac{c\Delta t}{Re}\left\langle \nabla\tilde{\rho}^{n+1}_h, \nabla\xi_h\right\rangle, \\ a_2\left(\rho^{n+1}_h, \xi_h\right) = \left\langle \rho^{n+1}_h, \xi_h\right\rangle + \frac{c\Delta t}{Re}\left\langle \nabla\rho^{n+1}_h, \nabla\xi_h\right\rangle, \\ l_1(\xi_h) = -\Delta t b_h\left(\rho^n_h, \xi_h, U^{\star,n+\frac{1}{2}}_h\right), \\ l_2(\xi_h) = -\frac{\Delta t}{2}\left(b_h\left(\rho^n_h, \xi_h, U^{\star,n+\frac{1}{2}}_h\right) + b_h\left(\tilde{\rho}^{n+1}_h, \xi_h, U^{\star,n+\frac{1}{2}}_h\right)\right) \end{cases} \quad (46)$$

We need to show that the bilinear forms a_1 and a_2 and the linear forms l_1 and l_2 satisfy the conditions of the Lax-Milgram theorem, i.e., a_1 and a_2 are continuous and coercive, and l_1 and l_2 are continuous.

For the coercivity of the bilinear forms a_1 and a_2, we can write:

$$\begin{aligned} a_1(\tilde{\rho}_h^{n+1}, \tilde{\rho}_h^{n+1}) &= \left\langle \tilde{\rho}_h^{n+1}, \tilde{\rho}_h^{n+1} \right\rangle + \frac{c\Delta t}{Re} \left\langle \nabla \tilde{\rho}_h^{n+1}, \nabla \tilde{\rho}_h^{n+1} \right\rangle \\ &= \|\tilde{\rho}_h^{n+1}\|^2_{L^2(Q)} + \frac{c\Delta t}{Re} \|\nabla \tilde{\rho}_h^{n+1}\|^2_{L^2(Q)} \\ &\geq \min\left(1, \frac{c\Delta t}{Re}\right) \|\tilde{\rho}_h^{n+1}\|^2_{H^1(Q)} \end{aligned} \tag{47}$$

Similarly,

$$\begin{aligned} a_2(\rho_h^{n+1}, \rho_h^{n+1}) &= \left\langle \rho_h^{n+1}, \rho_h^{n+1} \right\rangle + \frac{c\Delta t}{Re} \left\langle \nabla \rho_h^{n+1}, \nabla \tilde{\rho}_h^{n+1} \right\rangle \\ &= \|\rho_h^{n+1}\|^2_{L^2(Q)} + \frac{c\Delta t}{Re} \|\nabla \rho_h^{n+1}\|^2_{L^2(Q)} \\ &\geq \min\left(1, \frac{c\Delta t}{Re}\right) \|\rho_h^{n+1}\|^2_{H^1(Q)} \end{aligned} \tag{48}$$

This describes the coercivity of the bilinear forms a_1 and a_2.
For continuity, using the Cauchy-Schwarz inequality in the form:

$$(ac + bd)^2 \leq (a^2 + b^2)(c^2 + d^2),$$

where $a,\ b,\ c$ and d are positive real number:

$$\begin{aligned} |a_1(\tilde{\rho}_h^{n+1}, \xi_h)| &= \left| \left\langle \tilde{\rho}_h^{n+1}, \xi_h \right\rangle + \frac{c\Delta t}{Re} \left\langle \nabla \tilde{\rho}_h^{n+1}, \nabla \xi_h \right\rangle \right| \\ &\leq \int_Q |\tilde{\rho}_h^{n+1}||\xi_h| dx + \frac{c\Delta t}{Re} \int_Q |\nabla \tilde{\rho}_h^{n+1}||\nabla \xi_h| dx \\ &\leq C \|\tilde{\rho}_h^{n+1}\|_{H^1(Q)} \|\xi_h\|_{H^1(Q)} \end{aligned} \tag{49}$$

Similarly,

$$\begin{aligned} |a_2(\rho_h^{n+1}, \xi_h)| &= \left| \left\langle \rho_h^{n+1}, \xi_h \right\rangle + \frac{c\Delta t}{Re} \left\langle \nabla \rho_h^{n+1}, \nabla \xi_h \right\rangle \right| \\ &\leq \int_Q |\rho_h^{n+1}||\xi_h| dx + \frac{c\Delta t}{Re} \int_Q |\nabla \rho_h^{n+1}||\nabla \xi_h| dx \\ &\leq C \|\rho_h^{n+1}\|_{H^1(Q)} \|\xi_h\|_{H^1(Q)} \end{aligned} \tag{50}$$

where C is a constent dependent of $\dfrac{c\Delta t}{Re}$. We conclude that the bilinear forms a_1 and a_2 are continuous. For the continuity of the linear forms l_1 and l_2, we use Lemma 3.2, which essentially shows the continuity of the bilinear form b_h and, consequently, the continuity of the linear forms l_1 and l_2. We have:

$$\begin{aligned} |l_1(\xi_h)| &= \left| \Delta t b_h(\rho_h^n, \xi_h, U_h^{\star, n+\frac{1}{2}}) \right| \\ &\leq \Delta t C \|U_h^{\star, n+\frac{1}{2}}\|_{L^4(Q)} \|\nabla \rho_h^n\|_{L^4(Q)} \|\xi_h\|_{L^2(Q)} \\ &\leq \Delta t C \|U_h^{\star, n+\frac{1}{2}}\|_{L^4(Q)} \|\nabla \rho_h^n\|_{L^4(Q)} \|\Delta \xi_h\|_{L^2(Q)} \end{aligned} \tag{51}$$

and

$$\begin{aligned} |l_2(\xi_h)| &= \left| \frac{\Delta t}{2} \left(b_h(\rho_h^n, \xi_h, U_h^{\star, n+\frac{1}{2}}) + b_h(\tilde{\rho}_h^{n+1}, \xi_h, U_h^{\star, n+\frac{1}{2}}) \right) \right| \\ &\leq \frac{\Delta t}{2} C \left(\|U_h^{\star, n+\frac{1}{2}}\|_{L^4(Q)} \|\nabla \rho_h^n\|_{L^4(Q)} \|\xi_h\|_{L^2(Q)} \right) \\ &\quad + \frac{\Delta t}{2} C \left(\|U_h^{\star, n+\frac{1}{2}}\|_{L^4(Q)} \|\nabla \tilde{\rho}_h^{n+1}\|_{L^4(Q)} \|\xi_h\|_{L^2(Q)} \right) \\ &\leq \frac{\Delta t}{2} C \|U_h^{\star, n+\frac{1}{2}}\|_{L^4(Q)} \|\Delta \xi_h\|_{L^2(Q)} \left(\|\nabla \rho_h^n\|_{L^4(Q)} \right. \\ &\quad \left. + \|\nabla \tilde{\rho}_h^{n+1}\|_{L^4(Q)} \right) \end{aligned} \tag{52}$$

Obviously, relations (5) and (32) directly imply relation (45).
Therefore, the conditions required for the application of Lax-Milgram's theorem are established. So, there exists a unique solution $(\tilde{\rho}_h^{n+1}, \rho_h^{n+1})$ for the system (42) satisfying the pointwise estimates:

$$\begin{aligned} &0 < m \leq \rho_h^{n+1} \leq M < +\infty \\ \text{and} \quad &0 < m \leq \tilde{\rho}_h^{n+1} \leq M < +\infty. \end{aligned}$$

3.4. The hybrid finite volume-finite element scheme

The hybrid scheme combines finite volumes for mass conservation (two-step Runge-Kutta) and finite elements for momentum. The velocity temporal derivative uses an Euler scheme. Therefore, let's define the numerical scheme as follows:

Initialization: Let (ρ_h^0, U_h^0) be the approximation of (ρ^0, U^0) for h sufficiently small.
At time step n+1: Let $(\rho_h^n, U_h^n) \in \mathcal{W}_h \times \mathcal{V}_h$.

$$\begin{cases} \text{Find } (\tilde{\rho}_h^{n+1}, \rho_h^{n+1}) \in \mathcal{W}_h \times \mathcal{W}_h, \\ \text{such that } \forall \xi_h \in \mathcal{W}_h; \\ \left\langle \dfrac{\tilde{\rho}_h^{n+1} - \rho_h^n}{\Delta t}, \xi_h \right\rangle + b_h\left(\rho_h^n, \xi_h, U_h^{*,n+\frac{1}{2}}\right) \\ + \dfrac{c}{Re}\left\langle \nabla \tilde{\rho}_h^{n+1}, \nabla \xi_h \right\rangle = 0 \\ \left\langle \dfrac{\rho_h^{n+1} - \rho_h^n}{\Delta t}, \xi_h \right\rangle + \dfrac{1}{2}\left(b_h\left(\rho_h^n, \xi_h, U_h^{*,n+\frac{1}{2}}\right)\right. \\ \left. + b_h\left(\tilde{\rho}_h^{n+1}, \xi_h, U_h^{*,n+\frac{1}{2}}\right)\right) \\ + \dfrac{c}{Re}\left\langle \nabla \rho_h^{n+1}, \nabla \xi_h \right\rangle = 0 \\ \text{Let} \rho_h^{n+1} \in \mathcal{W}_h, \ \text{find } U_h^{n+1} \in \mathcal{V}_h, \\ \text{such that } \forall (\vartheta_h, \pi_h) \in \mathcal{V}_h \times \mathcal{V}_h : \\ \left\langle \rho_h^n \dfrac{U_h^{n+1} - U_h^n}{\Delta t}, \vartheta_h \right\rangle \\ + \dfrac{1}{2}\left(\dfrac{\rho_h^{n+1} - \rho_h^n}{\Delta t} U_h^{n+1}, \vartheta_h\right) \\ + a\left(\rho_h^{n+1}, U_h^{n+1}, \vartheta_h\right) \\ + c\left(\rho_h^{n+1} U_h^{n+1} - \dfrac{c}{Re} \nabla \rho_h^{n+1}, U_h^{n+1}, \vartheta_h\right) \\ - \left\langle \mathcal{P}_h^{n+1}, \nabla \vartheta_h \right\rangle = 0 \\ \left\langle \nabla \cdot U_h^{n+1}, \pi_h \right\rangle = 0 \\ U_h^{n+1}|_{\sum_T} = 0 \end{cases} \tag{53}$$

Equations (53) involve solving two linear systems to obtain $(\tilde{\rho}_h^{n+1}, \rho_h^{n+1}, U_h^{n+1})$. First, $(\tilde{\rho}_h^{n+1}, \rho_h^{n+1})$ is computed using finite volumes for the convection-diffusion equation, with $U_h^{*,n+\frac{1}{2}}$ as the velocity approximation. Then, U_h^{n+1} is solved with finite elements for the momentum conservation equation, ensuring incompressibility.
Now let $\Delta_h : \mathcal{W}_h \to \mathcal{W}_h$ be the linear form defined by:

$$-\left\langle \Delta_h \rho_h, \xi_h \right\rangle = \left\langle \nabla \rho_h, \nabla \xi_h \right\rangle, \quad \forall \xi_h \in \mathcal{W}_h.$$

Then, the finite volume scheme can be reformulated as follows:

$$\left\langle \frac{\tilde{\rho}_h^{n+1} - \rho_h^n}{\Delta t}, \xi_h \right\rangle + b_h\left(\rho_h^n, \xi_h, U_h^{*,n+\frac{1}{2}}\right) - \frac{c}{Re}\left\langle \Delta_h \tilde{\rho}_h^{n+1}, \xi_h \right\rangle = 0 \tag{54}$$

$$\left\langle \frac{\rho_h^{n+1} - \rho_h^n}{\Delta t}, \xi_h \right\rangle + \frac{1}{2}\left(b_h\left(\rho_h^n, \xi_h, U_h^{*,n+\frac{1}{2}}\right) + b_h\left(\tilde{\rho}_h^{n+1}, \xi_h, U_h^{*,n+\frac{1}{2}}\right)\right) - \frac{c}{Re}\left\langle \Delta_h \rho_h^{n+1}, \xi_h \right\rangle = 0 \tag{55}$$

In the next section, we prove the well-posedness of the discrete problem (53) and establish the discrete energy estimate for the hybrid scheme, independently of the discrete parameters.

4. Main results

Now, we give the main results of this paper.

Theorem 4.1. *Let $(\rho_{h,\Delta t}, U_{h,\Delta t})$ be a pair of discrete solutions of (53). When the parameters $(h, \Delta t)$ tend to zero, then $(\rho_{h,\Delta t}, U_{h,\Delta t})$ converges to (ρ, U) according to Definition 2.1, under the following conditions:*
$0 \le \lambda \le \dfrac{\mu}{4M}$ *and* $0 < \Delta t \le c_0 h$.

The following subsections are devoted to the proof of this main result.

4.1. Uniform estimates

Here, energy estimates for velocity and density projections are derived using the discrete Laplacian of density.
Throughout this section, the symbol C will denote a generic positive constant, which

may vary from line to line or from one estimate to another. Unless otherwise specified, this constant is independent of the discretization parameters (such as Δt, h, etc.).

Proposition 4.1. *There exists a unique solution* $(\tilde{\rho}_h^{n+1}, \rho_h^{n+1}, U_h^{n+1})$ *for the discrete problem* (53) *satisfying the following inequalities:*

$$\|\sqrt{\rho_h^{n+1}}U_h^{n+1}\|^2_{L^2(Q)} - \|\sqrt{\rho_h^n}U_h^n\|^2_{L^2(Q)} + \|\sqrt{\rho_h^n}(U_h^{n+1} - U_h^n)\|^2_{L^2(Q)} + 2\mu_1\|\nabla U_h^{n+1}\|^2_{L^2(Q)} \leq \mathcal{C}_3 \quad (56)$$

$$\|\nabla\tilde{\rho}_h^{n+1}\|^2_{L^2(Q)} - \|\nabla\rho_h^n\|^2_{L^2(Q)} + \|\nabla(\tilde{\rho}_h^{n+1} - \rho_h^n)\|^2_{L^2(Q)} + \|\Delta_h\tilde{\rho}_h^{n+1}\|^2_{L^2(Q)} \leq C_2\Delta t\|U_h^{*,n+\frac{1}{2}}\|^2_{L^2(Q)}\|\nabla U_h^{*,n+\frac{1}{2}}\|^2_{L^2(Q)}\|\nabla\rho_h^n\|^2_{L^2(Q)} + \frac{\Delta t}{2}\|\Delta\rho_h^n\|^2_{L^2(Q)} \quad (57)$$

$$\|\nabla\rho_h^{n+1}\|^2_{L^2(Q)} - \|\nabla\rho_h^n\|^2_{L^2(Q)} + \|\nabla(\rho_h^{n+1} - \rho_h^n)\|^2_{L^2(Q)} + \|\Delta_h\rho_h^{n+1}\|^2_{L^2(Q)} \leq C_4\Delta t\|U_h^{*,n+\frac{1}{2}}\|^2_{L^2(Q)}\|\nabla U_h^{*,n+\frac{1}{2}}\|^2_{L^2(Q)} \Big(\|\nabla\tilde{\rho}_h^{n+1}\|^2_{L^2(Q)} + \|\nabla\rho_h^n\|^2_{L^2(Q)}\Big) + \frac{\Delta t}{2}\Big(\|\Delta\tilde{\rho}_h^{n+1}\|^2_{L^2(Q)} + \|\Delta\rho_h^n\|^2_{L^2(Q)}\Big) \quad (58)$$

where Fr and Re are the Froude and Reynolds numbers respectively, M, C_2, $\mathcal{C}_3$, and C_4 are positive constants independent of $h, \Delta t$, and n.

Proof. For the proof of the first inequality (56), we take $\vartheta_h = 2\Delta t U_h^{n+1}$ in equation $(53)_3$, using the following equality

$$\langle a-b, 2a\rangle = \|a\|^2_{L^2(Q)} - \|b\|^2_{L^2(Q)} + \|a-b\|^2_{L^2(Q)},$$

and the properties of trilinear forms a and c, we obtain:

$$\|\sqrt{\rho_h^{n+1}}U_h^{n+1}\|^2_{L^2(Q)} - \|\sqrt{\rho_h^n}U_h^n\|^2_{L^2(Q)} + \|\sqrt{\rho_h^n}(U_h^{n+1} - U_h^n)\|^2_{L^2(Q)} + 2\alpha\Delta t\|\nabla U_h^{n+1}\|^2_{L^2(Q)} \leq 2\Delta t\left\langle \mathcal{P}_h^{n+1}, \nabla U_h^{n+1}\right\rangle \quad (59)$$

Moreover, for the right-hand side of (59), we use the discrete maximum principle, Hölder's, Young's and Poincaré's inequalities to obtain:

$$\begin{aligned}\left\langle \mathcal{P}_h^{n+1}, \nabla U_h^{n+1}\right\rangle &= \int_Q \Big[-\frac{c}{Re}(U_h^{n+1}.\nabla\rho_h^{n+1}) + \frac{1}{2Fr^2}(\rho_h^{n+1})^2\Big].\nabla U_h^{n+1}dx\\ &\leq \frac{2\Delta t c}{Re}\|\nabla U_h^{n+1}\|_{L^2(Q)}\|U_h^{n+1}\|_{L^4(Q)}\|\nabla\rho_h^{n+1}\|_{L^4(Q)} + \frac{2MC_1\Delta t}{Fr^2}\|\nabla\rho_h^{n+1}\|_{L^2(Q)}\|\nabla U_h^{n+1}\|_{L^2(Q)}\\ &\leq \frac{2\Delta t C_2 c}{Re}\|\nabla U_h^{n+1}\|_{L^2(Q)}\|U_h^{n+1}\|_{L^4(Q)}\|\nabla\rho_h^{n+1}\|_{L^4(Q)} + \frac{MC_1\Delta t}{Fr^2}\Big(\|\nabla\rho_h^{n+1}\|^2_{L^2(Q)} + \|\nabla U_h^{n+1}\|^2_{L^2(Q)}\Big)\end{aligned} \quad (60)$$

Then (59) becomes:

$$\|\sqrt{\rho_h^{n+1}}U_h^{n+1}\|^2_{L^2(Q)} - \|\sqrt{\rho_h^n}U_h^n\|^2_{L^2(Q)} + \|\sqrt{\rho_h^n}(U_h^{n+1} - U_h^n)\|^2_{L^2(Q)} + 2\alpha\Delta t\|\nabla U_h^{n+1}\|^2_{L^2(Q)} \leq \frac{2\Delta t C_2 c}{Re}\|\nabla U_h^{n+1}\|_{L^2(Q)}\|U_h^{n+1}\|_{L^4(Q)}\|\nabla\rho_h^{n+1}\|_{L^4(Q)} + \frac{MC_1\Delta t}{Fr^2}\Big(\|\nabla\rho_h^{n+1}\|^2_{L^2(Q)} + \|\nabla U_h^{n+1}\|^2_{L^2(Q)}\Big) \quad (61)$$

For the following of this proof, we set:

$$I_1 = \|\sqrt{\rho_h^{n+1}}U_h^{n+1}\|^2_{L^2(Q)} - \|\sqrt{\rho_h^n}U_h^n\|^2_{L^2(Q)} + \|\sqrt{\rho_h^n}(U_h^{n+1} - U_h^n)\|^2_{L^2(Q)}.$$

Using Gagliardo-Nirenberg inequality, a.e

$$\|U_h^{n+1}\|^2_{L^4(Q)} \leq C_3\|\nabla U_h^{n+1}\|_{L^2(Q)}\|U_h^{n+1}\|_{L^2(Q)},$$

Poincaré inequality and Sobolev embedding for velocity, the previous inequality gives:

$$I_1 + 2\alpha\Delta t\|\nabla U_h^{n+1}\|^2_{L^2(Q)}$$
$$\leq \frac{2\Delta t\mathcal{C}_1 c}{Re}\|\nabla U_h^{n+1}\|^2_{L^2(Q)}\|\nabla\rho_h^{n+1}\|_{L^4(Q)}$$
$$+\frac{MC_2\Delta t}{Fr^2}\left(\|\nabla\rho_h^{n+1}\|^2_{L^2(Q)} + \|\nabla U_h^{n+1}\|^2_{L^2(Q)}\right)$$
$$\leq 2\mathcal{C}_1\|\nabla U_h^{n+1}\|^2_{L^2(Q)}\|\nabla\rho_h^{n+1}\|_{L^4(Q)}$$
$$+\frac{MC_2\Delta t}{Fr^2}\left(\|\nabla\rho_h^{n+1}\|^2_{L^2(Q)} + \|\nabla U_h^{n+1}\|^2_{L^2(Q)}\right).$$

Which gives us:

$$I_1 + 2\|\nabla U_h^{n+1}\|^2_{L^2(Q)}\left(\alpha\Delta t - \mathcal{C}_1\|\nabla\rho_h^{n+1}\|_{L^4(Q)} - \frac{MC_2\Delta t}{2Fr^2}\right) \leq \frac{MC_2\Delta t}{Fr^2}\|\nabla\rho_h^{n+1}\|^2_{L^2(Q)} \quad (62)$$

Since $\rho_h^{n+1} \in \mathcal{W}_h \subset H^1(Q)$ and $H^1(Q) \hookrightarrow L^4(Q)$ then, inequality (62) becomes:

$$\|\sqrt{\rho_h^{n+1}}U_h^{n+1}\|^2_{L^2(Q)} - \|\sqrt{\rho_h^n}U_h^n\|^2_{L^2(Q)}$$
$$+\|\sqrt{\rho_h^n}(U_h^{n+1} - U_h^n)\|^2_{L^2(Q)} + 2\mu_1\|\nabla U_h^{n+1}\|^2_{L^2(Q)} \leq \mathcal{C}_3,$$

with $\quad \mu_1 = \alpha\Delta t - \mathcal{C}_1\mathcal{C}_2 - \dfrac{MC\Delta t}{2Fr^2}.$

Therefore, we deduce:

$$\|\sqrt{\rho_h^{n+1}}U_h^{n+1}\|^2_{L^2(Q)} - \|\sqrt{\rho_h^n}U_h^n\|^2_{L^2(Q)} + \|\sqrt{\rho_h^n}(U_h^{n+1} - U_h^n)\|^2_{L^2(Q)} + 2\mu_1\|\nabla U_h^{n+1}\|^2_{L^2(Q)} \leq \mathcal{C}_3 \quad (63)$$

This concludes the proof of inequality (56). For the second inequality, we take

$\xi_h = -\Delta t\Delta_h\tilde{\rho}_h^{n+1}$ in (54) to obtain:

$$\|\nabla\tilde{\rho}_h^{n+1}\|^2_{L^2(Q)} - \|\nabla\rho_h^n\|^2_{L^2(Q)} + \|\nabla(\tilde{\rho}_h^{n+1} - \rho_h^n)\|^2_{L^2(Q)} + \frac{2c\Delta t}{Re}\|\Delta_h\tilde{\rho}_h^{n+1}\|^2_{L^2(Q)} = 2\Delta t b_h\left(\rho_h^n, \Delta_h\tilde{\rho}_h^{n+1}, U_h^{*,n+\frac{1}{2}}\right) := I_2 \quad (64)$$

Using Lemmas 3.1 and 3.2, as well as the Young's inequality, we estimate the right-hand term I_2 in (64) as follows:

$$|I_2| \leq 2C_1\Delta t\|U_h^{*,n+\frac{1}{2}}\|_{L^4(Q)}\|\nabla\rho_h^n\|_{L^4(Q)}\|\Delta_h\tilde{\rho}_h^{n+1}\|_{L^2(Q)}$$
$$\leq \frac{c\Delta t}{Re}\|\Delta\tilde{\rho}_h^{n+1}\|^2_{L^2} + \frac{C_1 Re\Delta t}{c}\|U_h^{*,n+\frac{1}{2}}\|^2_{L^4(Q)}\|\nabla\rho_h^n\|^2_{L^4(Q)}$$
$$\leq \frac{c\Delta t}{Re}\|\Delta\tilde{\rho}_h^{n+1}\|^2_{L^2} + \frac{C_1 Re}{c}\Delta t\|U_h^{*,n+\frac{1}{2}}\|^2_{L^4(Q)}\|\nabla\rho_h^n\|_{L^2(Q)}\|\Delta\rho_h^n\|_{L^2(Q)} \quad (65)$$

Finally, using Young's inequality and Lemma 3.1 for the velocity, (65) becomes:

$$|I_2| \leq \frac{c\Delta t}{Re}\|\Delta\tilde{\rho}_h^{n+1}\|^2_{L^2(Q)} + \frac{c\Delta t}{2Re}\|\Delta\rho_h^n\|^2_{L^2(Q)} + C_2\Delta t\|U_h^{*,n+\frac{1}{2}}\|^2_{L^2(Q)}\|\nabla U_h^{*,n+\frac{1}{2}}\|^2_{L^2(Q)}\|\nabla\rho_h^n\|^2_{L^2(Q)} \quad (66)$$

Combining (64) and (65), we obtain (57). Following the same manage as for (57), we obtain (58) by taking $\xi_h = -\Delta t\Delta_h\rho_h^{n+1}$ in equation (55). Thus, we define:

$$I_3 = b_h\left(\tilde{\rho}_h^{n+1}, \Delta_h\rho_h^{n+1}, U_h^{*,n+\frac{1}{2}}\right);$$
$$I_4 = b_h\left(\rho_h^n, \Delta_h\rho_h^{n+1}, U_h^{*,n+\frac{1}{2}}\right).$$

Therefore, equivalently to inequality (66), we have:

$$|I_3| \leq \frac{c\Delta t}{2Re}\|\Delta\rho_h^{n+1}\|^2_{L^2(Q)} + \frac{c\Delta t}{2Re}\|\Delta\rho_h^n\|^2_{L^2(Q)} + C_3\Delta t\|U_h^{*,n+\frac{1}{2}}\|^2_{L^2(Q)}\|\nabla U_h^{*,n+\frac{1}{2}}\|^2_{L^2(Q)}\|\nabla\rho_h^n\|^2_{L^2(Q)} \quad (67)$$

$$|I_4| \leq \frac{c\Delta t}{2Re}\|\Delta\rho_h^{n+1}\|^2_{L^2(Q)} + \frac{c\Delta t}{2Re}\|\Delta\tilde{\rho}_h^{n+1}\|^2_{L^2(Q)} + C_4\Delta t\|U_h^{*,n+\frac{1}{2}}\|^2_{L^2(Q)}\|\nabla U_h^{*,n+\frac{1}{2}}\|^2_{L^2(Q)}\|\nabla\tilde{\rho}_h^{n+1}\|^2_{L^2(Q)} \quad (68)$$

By adding (67) and (68), we obtain:

$$|I_3| + |I_4| \leq \frac{c\Delta t}{Re}\|\Delta\rho_h^{n+1}\|^2_{L^2(Q)} + \frac{c\Delta t}{2Re}\left(\|\Delta\tilde{\rho}_h^{n+1}\|^2_{L^2(Q)} + \|\Delta\rho_h^n\|^2_{L^2(Q)}\right) + C_4\Delta t\|U_h^{*,n+\frac{1}{2}}\|^2_{L^2(Q)}\|\nabla U_h^{*,n+\frac{1}{2}}\|^2_{L^2(Q)}\left(\|\nabla\tilde{\rho}_h^{n+1}\|^2_{L^2(Q)} + \|\nabla\rho_h^n\|^2_{L^2(Q)}\right) \quad (69)$$

By associating (69) with the equivalent of (64), we obtain (58).

To establish the global stability of the scheme (53), we can derive estimates for the velocity and enhanced regularity for the density using Proposition 4.1. and discrete Gronwall's lemma.

Lemma 4.1. *Let $U_0 \in V$; $\rho \in H^1(Q)$, then the solution $(\tilde{\rho}_h^{n+1}, \rho_h^{n+1}, U_h^{n+1})$ for the discrete problem (53) satisfies the following estimates:*

$$(i) \max_{0\le n\le N} \|U_h^n\|_{L^2(Q)} \le C;$$
$$(ii) \sum_{n=0}^{N} \|\nabla U_h^n\|^2_{L^2(Q)} \le C;$$
$$(iii) \sum_{n=0}^{N-1} \|U_h^{n+1} - U_h^n\|^2_{L^2(Q)} \le C;$$
$$(iv) \max_{0\le n\le N} \left(\|\nabla \rho_h^n\|_{L^2(Q)}\right) \le C;$$
$$(v) \max_{0\le n\le N} \left(\|\nabla \tilde{\rho}_h^{n+1}\|_{L^2(Q)}\right) \le C;$$
$$(vi) \sum_{n=0}^{N-1} \Delta t\|\Delta \tilde{\rho}_h^{n+1}\|^2_{L^2(Q)} \le C;$$
$$(vii) \sum_{n=0}^{N-1} \Delta t\|\Delta \rho_h^{n+1}\|^2_{L^2(Q)} \le C;$$
$$(viii) \sum_{n=0}^{N-1} \|\tilde{\rho}_h^{n+1} - \rho_h^n\|^2_{L^2(Q)} \le C;$$
$$(ix) \sum_{n=0}^{N-1} \|\rho_h^{n+1} - \rho_h^n\|^2_{L^2(Q)} \le C,$$

where $C > 0$ is a constant depending on the initial conditions $(\rho_0, U_0, \frac{c}{Re})$ but independent of h, Δt and n.

Proof. Starting from the first inequality of Proposition 4.1., we have:

$$\|\sqrt{\rho_h^{n+1}}U_h^{n+1}\|^2_{L^2(Q)} - \|\sqrt{\rho_h^n}U_h^n\|^2_{L^2(Q)} + \|\sqrt{\rho_h^n}(U_h^{n+1} - U_h^n)\|^2_{L^2(Q)} + 2\mu_1\|\nabla U_h^{n+1}\|^2_{L^2(Q)} \le \mathcal{C}_3;$$

since $\rho_h^{n+1} \le M$ and U_h^{n-1}, U_h^n are known and in $H_0^1(Q)$, by summing from 0 to $N-1$, we recover estimates (i),(ii), and (iii).

For the remaining estimates, we start from the second inequality of Proposition 4.1. Considering the characterization of the velocity $U^{\star,n+\frac{1}{2}}$ in equation (32) and the properties of the density, we can rewrite this inequality as follows:

$$\|\nabla\tilde{\rho}_h^{n+1}\|^2_{L^2(Q)} - \|\nabla\rho_h^n\|^2_{L^2(Q)} + \|\nabla(\tilde{\rho}_h^{n+1} - \rho_h^n)\|^2_{L^2(Q)} + \Delta t\|\Delta_h\tilde{\rho}_h^{n+1}\|^2_{L^2(Q)} \le C_2\Delta t\|U_h^{*,n+\frac{1}{2}}\|^2_{L^2(Q)}\|\nabla U_h^{*,n+\frac{1}{2}}\|^2_{L^2(Q)}\|\nabla\rho_h^n\|^2_{L^2(Q)} + \frac{1}{2}\|\Delta\rho_h^n\|^2_{L^2(Q)} \quad (70)$$

Thus, with $U_h^{*,n+\frac{1}{2}}$ and $\nabla U_h^{*,n+\frac{1}{2}}$ in $H_0^1(Q)$; $\|\Delta\rho_h\|_{L^2(Q)} \le \hat{c}\|\rho_h\|_{L^2(Q)}$ and $\rho_h^{n+1} \le M$; inequality (70) becomes:

$$\|\nabla\tilde{\rho}_h^{n+1}\|^2_{L^2(Q)} - \|\nabla\rho_h^n\|^2_{L^2(Q)} + \|\nabla(\tilde{\rho}_h^{n+1} - \rho_h^n)\|^2_{L^2(Q)} + \Delta t\|\Delta_h\tilde{\rho}_h^{n+1}\|^2_{L^2(Q)} \le C \quad (71)$$

Summing (71) from, 0 to $N-1$ and using Gronwall's lemma, we obtain estimates (iv), (vi) and $(viii)$. Following the same process and using estimates (iv) and (vi), we also find estimates (v), (vii) and (ix).

Finally, based on Lemma 4.1, we deduce the following result:

Corollary 4.1. *Under the assumptions of Lemma 4.1, the following estimates hold:*

$$\sum_{n=0}^{N} \|\nabla\rho_h^n\|^4_{L^4(Q)} \le C \text{ and } \sum_{n=0}^{N} \|\nabla\tilde{\rho}_h^{n+1}\|^4_{L^4(Q)} \le C,$$

where $C > 0$ is independent of h and Δt.

Proof. This result is a direct consequence of Theorem 3.1 and Lemma 3.1.

4.2. Weak convergence

To study the convergence of the hybrid finite volume-finite element scheme (53) to the weak solution of (3)-(4), we define the following auxiliary functions.

In the following, we introduce the piecewise constant-in-time functions defined for $t_n < t \leq t_{n+1}$:

$$U_{h,\Delta t}(t) = U_h^{n+1},\ \widehat{U}_{h,\Delta t}(t) = U_h^n,\ \widetilde{U}_{h,\Delta t}(t) = U_h^{\star,n+\frac{1}{2}},$$

$$\rho_{h,\Delta t}(t) = \rho_h^{n+1},\ \widetilde{\rho}_{h,\Delta t}(t) = \widetilde{\rho}_h^{n+1},\ \widehat{\rho}_{h,\Delta t}(t) = \rho_h^n.$$

In addition, we define $\breve{\rho}_{h,\Delta t} \in \mathcal{C}([0,T],\mathcal{W}_h)$ as the piecewise linear (continuous-in-time) function given on each interval $[t_n, t_{n+1}]$ by:

$$\breve{\rho}_{h,\Delta t}(t) = \rho_h^{n+1} + \frac{\rho_h^{n+1} - \rho_h^n}{\Delta t}(t - t_{n+1}).$$

Using Lemma 4.1 and Corollary 4.1, we arrive at the following result (see [3], [6] and [7]):

Lemma 4.2. *The following estimates (independent of h and Δt) hold:*

$$\{U_{h,\Delta t}\}_{h,\Delta t}, \{\widehat{U}_{h,\Delta t}\}_{h,\Delta t}, \{\widetilde{U}_{h,\Delta t}\}_{h,\Delta t} \text{ are bounded in } L^\infty(0,T;L^2(Q)) \cap L^2(0,T;H_0^1(Q)) \quad (72)$$

$$\{\rho_{h,\Delta t}\}_{h,\Delta t}, \{\widehat{\rho}_{h,\Delta t}\}_{h,\Delta t}, \{\widetilde{\rho}_{h,\Delta t}\}_{h,\Delta t} \text{ are bounded in } L^\infty(0,T;H^1(Q)) \cap L^\infty(Q_T) \cap L^2(0,T;H^2(Q)) \quad (73)$$

$$\{\nabla\rho_{h,\Delta t}\}_{h,\Delta t}, \{\nabla\widetilde{\rho}_{h,\Delta t}\}_{h,\Delta t} \text{ are bounded in } L^4(0,T;L^4(Q)) \quad (74)$$

Also,

$$\|U_{h,\Delta t} - \widehat{U}_{h,\Delta t}\|^2_{L^2(0,T;L^2(Q))} \leq C\Delta t \quad \text{and}$$
$$\|\rho_{h,\Delta t} - \widehat{\rho}_{h,\Delta t}\|^2_{L^2(0,T;H^1(Q))} \leq C\Delta t,$$
$$\|\widetilde{\rho}_{h,\Delta t} - \widehat{\rho}_{h,\Delta t}\|^2_{L^2(0,T;H^1(Q))} \leq C\Delta t \quad (75)$$

Thanks to the a priori estimates of Lemma 4.1, and in view of the bounds established in Lemma 4.2, we know that the families $\{U_{h,\Delta t}\}_{h,\Delta t}, \{\widehat{U}_{h,\Delta t}\}_{h,\Delta t}, \{\widetilde{U}_{h,\Delta t}\}_{h,\Delta t}, \{\rho_{h,\Delta t}\}_{h,\Delta t}, \{\widehat{\rho}_{h,\Delta t}\}_{h,\Delta t}, \{\widetilde{\rho}_{h,\Delta t}\}_{h,\Delta t}$ are uniformly bounded in the appropriate functional spaces. By weak compactness, we may thus extract sub-sequences, still denoted in the same way, that converge weakly to some limit functions $U,\ \widehat{U},\ \widetilde{\rho},\ \rho,\ \widehat{\rho}$. Moreover, from the discrete relation (75), the limits satisfy $U = \widehat{U},\ \widetilde{\rho} = \widehat{\rho},:\ \rho = \widehat{\rho}$. We can therefore state the following result (see [7, Lemma 5.3] for the proof).

Lemma 4.3. *There exists sub-sequences of $\{U_{h,\Delta t}\}_{h,\Delta t}, \{\widehat{U}_{h,\Delta t}\}_{h,\Delta t}, \{\rho_{h,\Delta t}\}_{h,\Delta t}, \{\widehat{\rho}_{h,\Delta t}\}_{h,\Delta t}, \{\widetilde{\rho}_{h,\Delta t}\}_{h,\Delta t}, \{\nabla\rho_{h,\Delta t}\}_{h,\Delta t}$ and $\{\nabla\widetilde{\rho}_{h,\Delta t}\}_{h,\Delta t}$ and functions $U,\ \rho$ satisfying the following weak convergences, as $(h,\Delta t) \to (0,0)$:*

$$U_{h,\Delta t} \to U,\ \widehat{U}_{h,\Delta t} \to U,\ \widetilde{U}_{h,\Delta t} \to U$$
$$\text{in } \begin{cases} L^2(0,T;H_0^1(Q)) - \text{weakly} \\ L^\infty(0,T;L^2(Q)) - \text{weakly} - \star \end{cases} \quad (76)$$

$$\rho_{h,\Delta t} \to \rho,\ \widehat{\rho}_{h,\Delta t} \to \rho,\ \widetilde{\rho}_{h,\Delta t} \to \rho,\ \breve{\rho} \to \rho$$
$$\text{in } \begin{cases} L^\infty(0,T;H^1(Q)) - \text{weakly} - \star \\ L^\infty(Q_T) - \text{weakly} - \star \\ L^2(0,T;H^2(Q)) - \text{weakly} \end{cases} \quad (77)$$

$$\nabla\rho_{h,\Delta t} \to \nabla\rho,\ \nabla\widetilde{\rho} \to \nabla\rho$$
$$\text{in } L^4(0,T;L^4(Q)) \quad (78)$$

4.3. Strong convergence

For nonlinear systems, strong convergence of the hybrid scheme (53) is key to handling nonlinear terms. We establish compactness for both discrete density and velocity.

Proposition 4.2. *The following estimates hold:*

$$(i) \sum_{n=0}^{N-1} \|\widetilde{\rho}_h^{n+1} - \rho_h^n\|^{4/3}_{L^2(Q)} \leq C,$$
$$(ii) \sum_{n=0}^{N-1} \|\rho_h^{n+1} - \rho_h^n\|^{4/3}_{L^2(Q)} \leq C,$$

where $C > 0$ is a constant depending on the initial conditions (ρ_0, U_0) but independent of h, Δt and n.

Proof. For estimate (i), we follow the proof of Proposition 4 in [10]. In (54), we use Lemma 3.2, the Cauchy-Schwarz inequality, and the Sobolev embedding $H_0^1(Q) \subset L^4(Q)$ for all $\xi_h \in \mathcal{W}_h$.

$$\left|\left\langle \frac{\tilde{\rho}_h^{n+1} - \rho_h^n}{\Delta t}, \xi_h \right\rangle\right| \leq C \|\nabla U^{\star,n+\frac{1}{2}}\|_{L^2(Q)} \|\nabla \rho_h^n\|_{L^4(Q)} \|\xi\|_{L^2(Q)} + \frac{c}{Re} \|\Delta_h \tilde{\rho}_h^{n+1}\|_{L^2(Q)} \|\xi_h\|_{L^2(Q)}$$

$$\left|\left\langle \frac{\tilde{\rho}_h^{n+1} - \rho_h^n}{\Delta t}, \xi_h \right\rangle\right| \leq \Big(C \|\nabla U^{\star,n+\frac{1}{2}}\|_{L^2(Q)} \|\nabla \rho_h^n\|_{L^4(Q)} + \frac{c}{Re} \|\Delta_h \tilde{\rho}_h^{n+1}\|_h \Big) \|\xi\|_h \quad (79)$$

Knowing that $\dfrac{\tilde{\rho}_h^{n+1} - \rho_h^n}{\Delta t} \in \mathcal{W}_h \subset L^2(Q)$ and using continuity, from (79) we deduce:

$$\left\|\frac{\tilde{\rho}_h^{n+1} - \rho_h^n}{\Delta t}\right\|_{L^2(Q)} \leq C \|\nabla U^{\star,n+\frac{1}{2}}\|_{L^2(Q)} \|\nabla \rho_h^n\|_{L^4(Q)} + \frac{c\hat{c}_2}{Re} \|\Delta_h \tilde{\rho}_h^{n+1}\|_{L^2(Q)} \quad (80)$$

Using the Minkowski inequality and summing (80) for $n = \overline{0, N-1}$, we obtain:

$$\sum_{n=0}^{N-1} \left\|\frac{\tilde{\rho}_h^{n+1} - \rho_h^n}{\Delta t}\right\|_{L^2(Q)}^{4/3} \leq C \sum_{n=0}^{N-1} \|\nabla U^{\star,n+\frac{1}{2}}\|_{L^2(Q)}^{4/3} \|\nabla \rho_h^n\|_{L^4(Q)}^{4/3} + C \left(\frac{c}{Re}\right)^{4/3} \sum_{n=0}^{N-1} \|\Delta_h \tilde{\rho}_h^{n+1}\|_{L^2(Q)}^{4/3}.$$

Finally, applying the Hölder inequality, the estimates provided by Lemma 4.1 and Corollary 4.1, we directly obtain the desired result, i.e., estimate (i).
For estimate (ii), following the same manage as in the proof of (i), we obtain the following inequality, similar to (80):

$$\left\|\frac{\rho_h^{n+1} - \rho_h^n}{\Delta t}\right\|_{L^2(Q)} \leq \Big(C_1 \|\nabla U^{\star,n+\frac{1}{2}}\|_{L^2(Q)} \|\nabla \rho_h^n\|_{L^4(Q)} + \frac{c\hat{c}_2}{Re} \|\Delta_h \tilde{\rho}_h^{n+1}\|_{L^2(Q)} \Big) + \Big(C_2 \|\nabla U^{\star,n+\frac{1}{2}}\|_{L^2(Q)} \|\nabla \tilde{\rho}_h^{n+1}\|_{L^4(Q)} + \frac{c\hat{c}_2}{Re} \|\Delta_h \rho_h^{n+1}\|_{L^2(Q)} \Big) \quad (81)$$

Thus, we have, by using the same arguments as for (i):

$$\sum_{n=0}^{N-1} \left\|\frac{\rho_h^{n+1} - \rho_h^n}{\Delta t}\right\|_{L^2(Q)}^{4/3} \leq C_1 \sum_{n=0}^{N-1} \|\nabla U^{\star,n+\frac{1}{2}}\|_{L^2(Q)}^{4/3} \|\nabla \rho_h^n\|_{L^4(Q)}^{4/3} + \left(\frac{c}{Re}\right)^{4/3} \sum_{n=0}^{N-1} \|\Delta_h \tilde{\rho}_h^{n+1}\|_{L^2(Q)}^{4/3} + \left(\frac{c}{Re}\right)^{4/3} \sum_{n=0}^{N-1} \|\Delta_h \rho_h^{n+1}\|_{L^2(Q)}^{4/3} + C_2 \sum_{n=0}^{N-1} \|\nabla U^{\star,n+\frac{1}{2}}\|_{L^2(Q)}^{4/3} \|\nabla \tilde{\rho}_h^{n+1}\|_{L^4(Q)}^{4/3} \quad (82)$$

With (82), we can conclude the estimate (ii).

Corollary 4.2. *From Proposition 4.2, we deduce the following estimate:*

$$\left\|\frac{d}{dt} \breve{\rho}_{h,\Delta t}\right\|_{L^2(0,T;L^2(Q))}^{4/3} \leq C \quad (83)$$

From (83), we can derive the following strong convergences for the density, thanks to a compactness theorem of Aubin-Lions type (see [13]):

$$\rho_{h,\Delta t} \to \rho, \ \ \widehat{\rho}_{h,\Delta t} \to \rho, \widetilde{\rho}_{h,\Delta t} \to \rho, \breve{\rho}_{h,\Delta t} \to \rho \text{ in } L^2(0,T;L^2(Q))\text{-strongly as } (h,\Delta t) \to 0 \quad (84)$$

Moreover, according to Lemma 4.1, the discrete density is bounded in $L^\infty(Q_T)$, so we also obtain strong convergence in $L^p(Q_T)$ for $p < \infty$. For $p = \infty$, we can deduce convergence for at least a sub-sequence of $\rho_{h,\Delta t}$, $\widetilde{\rho}_{h,\Delta t}$, or $\widehat{\rho}_{h,\Delta t}$ to ρ almost everywhere in $(t,x) \in Q_T$.

We now estimate the fractional temporal derivative for the discrete velocity.

Proposition 4.3. *The following estimate holds:*

$$\int_0^{T-\delta} \left\| \sqrt{\rho_{h,\Delta t}(t+\delta)} \left(U_{h,\Delta t}(t+\delta) - U_{h,\Delta t}(t) \right) \right\|_{L^2(Q)}^2 dt \leq C \delta^{1/2} \quad (85)$$
$$\forall \delta : 0 < \delta < T,$$

where $C > 0$ is independent of $(h, \Delta t, \delta)$.

Proof. Since $\rho_{h,\Delta t}$ and $U_{h,\Delta t}$ are piecewise constant functions, to obtain (85) it suffices to consider δ as a multiple of the time step Δt, that is $\delta = r\Delta t$ with $r = 0, \cdots, N$ and to prove

$$\Delta t \sum_{m=0}^{N-r} \left\| \sqrt{\rho_h^{m+r}} \left(U_h^{m+r} - U_h^m \right) \right\|_{L^2(Q)}^2 \leq C(r\Delta t)^{1/2} \tag{86}$$

Firstly, we write the time derivative to the momentum equation $(53)_3$ in a conservative form. It is obtained by adding to both sides of $(53)_3$ the term,

$$\frac{1}{2} \left\langle \frac{\rho_h^{n+1} - \rho_h^n}{\Delta t} U_h^{n+1}, \vartheta_h \right\rangle.$$

So we obtain the following from $(53)_3$ by taking as test function $\vartheta_h = U_h^{m+r} - U_h^m$, using this identity :

$$\begin{aligned} &\left\langle \rho_n^{m+r} U_h^{m+r} - \rho_h^m U_h^m, U_h^{m+r} - U_h^m \right\rangle \\ &= \left\langle \rho_h^{m+r} \left\langle U_h^{m+r} - U_h^m \right\rangle, U_h^{m+r} - U_h^m \right\rangle \\ &\quad + \left\langle \left(\rho_h^{m+r} - \rho_h^m \right) U_h^m, U_h^{m+r} - U_h^m \right\rangle, \end{aligned}$$

multiplying by Δt and summing for $n = m, \cdots, m-1+r$:

$$\begin{aligned} &\left\| \sqrt{\rho_h^{m+r}} \left(U_h^{m+r} - U_h^m \right) \right\|_{L^2(Q)}^2 \\ &= -\left\langle \left(\rho_h^{m+r} - \rho_h^m \right) U_h^m, U_h^{m+r} - U_h^m \right\rangle \\ &\quad -\Delta t \sum_{n=m}^{m-1+r} c \left(\rho_h^{n+1} U_h^{n+1} - \frac{c}{Re} \nabla \rho_h^{n+1}, U_h^{n+1}, \vartheta_h \right) \\ &\quad +\Delta t \sum_{n=m}^{m-1+r} \left\langle \mathcal{P}_h^{n+1}, \nabla \vartheta_h \right\rangle \\ &\quad -\Delta t \sum_{n=m}^{m-1+r} a \left(\rho_h^{n+1}, U_h^{n+1}, \vartheta_h \right) \\ &\quad +\frac{1}{2} \Delta t \sum_{n=m}^{m-1+r} \left\langle \frac{\rho_h^{n+1} - \rho_h^n}{\Delta t} U_h^{n+1}, \vartheta_h \right\rangle \end{aligned} \tag{87}$$

On the other hand summing for $n = m, \cdots, m-r+1$, multiplying by Δt and taking $\xi_h = \rho_h^m - \rho_h^{m+r}$ as a test function in the density equation $(53)_2$, we obtain the following, using the properties of the trilinear form b, as well as Hölder's and Cauchy-Schwarz's inequalities.

$$\begin{aligned} \left| \rho_h^m - \rho_h^{m+r} \right|_h^2 &\leq \frac{\Delta t}{2} \sum_{n=m}^{m-1+r} \Big(\|U_h^{\star,n+\frac{1}{2}}\|_{L^4(Q)} \|\nabla \rho_h^n\|_{L^4(Q)} \\ &\quad \|\rho_h^m - \rho_h^{m+r}\|_{L^2(Q)} \Big) + \frac{\Delta t}{2} \sum_{n=m}^{m-1+r} \Big(\|U_h^{\star,n+\frac{1}{2}}\|_{L^4(Q)} \\ &\quad \|\nabla \tilde{\rho}_h^{n+1}\|_{L^4(Q)} \, \|\rho_h^m - \rho_h^{m+r}\|_{L^2(Q)} \Big) + \frac{2c}{Re} \sum_{n=m}^{m-1+r} \Big(\\ &\quad \|\Delta_h \rho_h^{n+1}\|_{L^2(Q)} \|\rho_h^m - \rho_h^{m+r}\|_{L^2(Q)} \Big). \end{aligned}$$

$$\begin{aligned} \|\rho_h^m - \rho_h^{m+r}\|_{L^2(Q)} &\leq \frac{C\Delta t}{2} \sum_{n=m}^{m-1+r} \Big(\|U_h^{\star,n+\frac{1}{2}}\|_{L^4(Q)} \, \|\nabla \rho_h^n\|_{L^4(Q)} \Big) \\ &+ \frac{C\Delta t}{2} \sum_{n=m}^{m-1+r} \|U_h^{\star,n+\frac{1}{2}}\|_{L^4(Q)} \|\nabla \tilde{\rho}_h^{n+1}\|_{L^4(Q)} + \frac{2c\Delta t}{Re} \sum_{n=m}^{m-1+r} \|\Delta_h \rho_h^{n+1}\|_{L^2(Q)} \\ &\leq \frac{C}{2} \Bigg(\Delta t \sum_{n=m}^{m-1+r} \|U_h^{\star,n+\frac{1}{2}}\|_{L^4(Q)} \|\nabla \rho_h^n\|_{L^4(Q)} + \|U_h^{\star,n+\frac{1}{2}}\|_{L^4(Q)} \|\nabla \tilde{\rho}_h^{n+1}\|_{L^4(Q)} \\ &\quad + \frac{c\Delta t}{Re} \sum_{n=m}^{m-1+r} \|\Delta_h \rho_h^{n+1}\|_{L^2(Q)} \Bigg)^{1/2} \left(\sum_{n=m}^{m-1+r} \Delta t \right)^{1/2} \\ &\leq \frac{C(r\Delta t)^{1/2}}{2} \leq C(r\Delta t)^{1/2}. \end{aligned}$$

Therefore,

$$\max_{1 \leq m \leq N} \|\rho_h^m - \rho_h^{m+r}\|_{L^2(Q)} \leq C(r\Delta t)^{1/2} \tag{88}$$

Multiplying (87) by Δt, summing for $m = 0, \cdots, N-r$ and bounding adequately, we can obtain the required bound (86). So we have:

$$\Delta t \sum_{m=0}^{N-r} \left\| \sqrt{\rho_h^{m+r}} \left(U_h^{m+r} - U_h^m \right) \right\|_{L^2(Q)}^2 \leq \sum_{i=1}^{5} |J_i| \tag{89}$$

Now, we show that each $|J_i|$ can be increased by $C(r\Delta t)^{1/2}$. In the following, we will prove J_1 and J_4; the others are handled in the same way. For more details, the reader may refer to [3]. For J_1, we use (88) to obtain:

$$\begin{aligned} J_1 &\leq \Delta t \sum_{m=0}^{N-r} \|\rho_h^m - \rho_h^{m+r}\|_{L^2(Q)} \|U_h^m\|_{L^4(Q)} \|U_h^{m+r} - U_h^m\|_{L^4(Q)} \\ &\leq \max_{1\leq m\leq N} \|\rho_h^m - \rho_h^{m+r}\|_{L^2(Q)} \left(\Delta t \sum_{m=0}^{N-r} \|U_h^m\|^2_{L^4(Q)}\right)^{1/2} \\ &\quad \times \left(\Delta t \sum_{m=0}^{N-r} \|U_h^{m+r} - U_h^m\|^2_{L^4(Q)}\right)^{1/2} \\ &\leq C(r\Delta t)^{1/2} \end{aligned} \tag{90}$$

For J_4 we have:

$$\begin{aligned} J_4 &\leq C(\Delta t)^2 \sum_{m=0}^{N-r} \sum_{n=m}^{m-1+r} \left(\frac{c}{Re} \|U_h^{n+1}\|_{L^4(Q)} \right. \\ &\quad \times \|\nabla \rho_h^{n+1}\|_{L^4(Q)} \|U_h^{m+r} - U_h^m\|_{L^2(Q)} \\ &\quad \left. + \frac{M}{Fr^2} \|\nabla \rho_h^{n+1}\|_{L^2(Q)} \|U_h^{m+r} - U_h^m\|_{L^2(Q)} \right) \\ &\leq C\Delta t \sum_{m=0}^{N-r} \left(\frac{c}{Re} \|U_h^{n+1}\|_{L^4(Q)} \|\nabla \rho_h^{n+1}\|_{L^4(Q)} \right. \\ &\quad \left. + \frac{M}{Fr^2} \|\nabla \rho_h^{n+1}\|_{L^2(Q)} \right) \times \left(\sum_{m=\widetilde{n-r+1}}^{\tilde{n}} \Delta t \|U_h^{m+r} \right. \\ &\quad \left. - U_h^m\|^2_{L^2(Q)} \right)^{1/2} \times \left(\sum_{m=\widetilde{n-r+1}}^{\tilde{n}} \Delta t \right)^{1/2} \\ &\leq C(r\Delta t)^{1/2} \left(\sum_{m=0}^{N-r} \left(\Delta t \frac{c}{Re} \|U_h^{n+1}\|_{L^4(Q)} \right. \right. \\ &\quad \left. \left. \|\nabla \rho_h^{n+1}\|_{L^4(Q)} + \frac{M}{Fr^2} \|\nabla \rho_h^{n+1}\|_{L^2(Q)} \right)^2 \right)^{1/2} \\ &\quad \times \left(\sum_{m=0}^{N-r} \Delta t \right)^{1/2} \leq C(r\Delta t)^{1/2}. \end{aligned}$$

with

$$\tilde{n} = \begin{cases} 0 & \text{if} \quad n < 0, \\ n & \text{if} \quad 0 \leq n \leq N-r, \\ N-r & \text{if} \quad n > N-r, \end{cases}$$

and $|\tilde{n} - \widetilde{n - r + 1}| \leq r$. This completes the proof of Proposition 4.3.

Finally, based on the result and thanks to the Aubin-Lions type compactness (see [[16], Theorem 5]), we obtain the following strong convergence for the velocity.

Corollary 4.3. *Let $U_{h,\Delta t}$ and $\widehat{U}_{h,\Delta t}$ be the piecewise constant functions taking values in U_h^{n+1} and U_h^n respectively. Following the previous proposition, we have the following convergences, as $(h, \Delta t) \to (0, 0)$:*

$$U_{h,\Delta t} \to U, \ \widehat{U}_{h,\Delta t} \to U \quad \textit{in} \quad L^2(0,T; L^2(Q))\textit{-strongly} \tag{91}$$

4.4. Passing to the limit

We aim to show that the approximate solution $(\tilde{\rho}_h^{n+1}, \rho_h^{n+1}, U_h^{n+1})$, obtained from the hybrid finite volume-finite element scheme (53), converges to the weak solution (ρ, U) of the Kazhikhov-Smagulov model (3)-(4) as h and Δt tend to zero.

For mass conservation, the limit transition for the finite volume scheme (42) as $(h, \Delta t) \to (0, 0)$ is detailed by Feistauer *et al.* (see [[17]], section E). However, in our case, the velocity in the convective part depends on $(h, \Delta t)$. To address this, we use the following strong-weak convergence result.

Lemma 4.4. *(See [10]) Let $(v_n)_n \in L^2(0,T; L^2(Q))$ and $(\eta_n)_n \in L^2(0,T; H^1(Q))$ such that $v_n \to v$ in $L^2(0,T; L^2(Q))-$strongly and $\eta_n \to \eta$ on $L^2(0,T; H^1(Q))-$weakly, then for any $\phi \in \mathcal{C}^1([0;T]; H^1(Q))$, such that $\phi(.,T) = 0$, we have:*

$$\int_0^T \int_Q v_n.\nabla \eta_n \phi dxdt \to \int_0^T \int_Q v.\nabla \eta \phi dxdt \quad \textit{as } n \to \infty. \tag{92}$$

To take the limit in the momentum conservation equation, we select a suitable test function $v \in \mathcal{C}^1([0,T]; \mathcal{C}_c^\infty(Q))$ such that $\nabla \cdot v = 0$ and $v(T, \cdot) = 0$. We define v_h^n as the projection of $v(t^n)$ onto $\mathcal{V}_h$. Let $v_{h,\Delta t} \in L^\infty(0,T; \mathcal{V}_h)$ be the piecewise constant function taking the value v_h^{n+1} on

$(t^n, t^{n+1}]$, and let $v_{h,\Delta t} \in \mathcal{C}^0([0,T]; \mathcal{V}_h)$ be the continuous piecewise linear function such that $\widetilde{v}_{h,\Delta t}(t^n) = v_h^n$. Then, as $(h, \Delta t) \to 0$, we have:

$$\begin{aligned} v_{h,\Delta t} &\to v \quad in \quad L^\infty(0,T; H_0^1(Q)), \\ \widetilde{v}_{h,\Delta} &\to v \quad in \quad W^{1,\infty}(0,T; H_0^1(Q)). \end{aligned}$$

Now, we are going to write the time derivative of the discrete equation $(53)_3$ in the conservative form.

By adding the following term to the right and the left-hand sides of $(53)_3$,

$$\frac{1}{2}\left\langle \frac{\rho_h^{n+1} - \rho_h^n}{\Delta t} U_h^{n+1}, \vartheta_h \right\rangle,$$

we obtain,

$$\begin{aligned} &\left\langle \frac{\rho_h^{n+1} U_h^{n+1} - \rho_h^n U_h^n}{\Delta t}, \vartheta_h \right\rangle + c\left(\rho_h^{n+1} U_h^{n+1} \right. \\ &\left. -\frac{c}{Re}\nabla \rho_h^{n+1}, U_h^{n+1}, \vartheta_h\right) + a\left(\rho_h^{n+1}, U_h^{n+1}, \vartheta_h\right) \\ &= \left(\mathcal{P}_h^{n+1}, \nabla\vartheta_h\right) + \frac{1}{2}\left\langle \frac{\rho_h^{n+1} - \rho_h^n}{\Delta t} U_h^{n+1}, \vartheta_h \right\rangle \end{aligned} \tag{93}$$

Next, taking $\vartheta_h = v_h^{n+1}$ as test function in (93), multiplying by Δt, summing for $n = 0, \cdots, N-1$, and using the following discrete integration by parts in time,

$$\begin{aligned} \sum_{n=0}^{N-1} \left\langle \rho_h^{n+1} U_h^{n+1} - \rho_h^n U_h^n, v_h^{n+1} \right\rangle &= \left\langle \rho_h^N U_h^N, v_h^N \right\rangle \\ - \sum_{n=0}^{N-1} \left\langle \rho_h^n U_h^n, v_h^{n+1} - v_h^n \right\rangle &- \left\langle \rho_h^0 U_h^0, v_h^0 \right\rangle, \end{aligned}$$

with $v_h^N = 0$; (since $v(T,.) = 0$), we arrive to the following conservative form:

$$\begin{aligned} &-\Delta t \sum_{n=0}^{N-1} \left\langle \rho_h^n U_h^n, \frac{v_h^{n+1} - v_h^n}{\Delta t} \right\rangle + \Delta t \sum_{n=0}^{N-1} a \\ &\times \left(\rho_h^{n+1}, U_h^{n+1}, v_h^{n+1}\right) + \Delta t \sum_{n=0}^{N-1} c\left(\rho_h^{n+1} U_h^{n+1}\right. \\ &\left. -\frac{c}{Re}\nabla\rho_h^{n+1}, U_h^{n+1}, v_h^{n+1}\right) - \left\langle \rho_h^0 U_h^0, v_h^0 \right\rangle \\ &= \Delta t \sum_{n=0}^{N-1} \left\langle \mathcal{P}_h^{n+1}, \nabla v_h^{n+1} \right\rangle \\ &\quad + \frac{\Delta t}{2} \sum_{n=0}^{N-1} \left\langle \frac{\rho_h^{n+1} - \rho_h^n}{\Delta t} U_h^{n+1}, v_h^{n+1} \right\rangle \end{aligned} \tag{94}$$

Then, by using the definitions introduced at the beginning of Section 4.2, we obtain the following variational formulation:

$$\begin{aligned} &-\int_0^T \left\langle \widehat{\rho}_{h,\Delta t} \widehat{U}_{h,\Delta t}, \frac{\partial}{\partial t}\widetilde{v}_{h,\Delta t} \right\rangle - \left\langle \rho_h^0 U_h^0, v_h^0 \right\rangle \\ &+ \int_0^T a\left(\rho_{h,\Delta t}, U_{h,\Delta t}, v_{h,\Delta t}\right) + \int_0^T c\left(\rho_{h,\Delta t} U_{h,\Delta t}\right. \\ &\left. -\frac{c}{Re}\nabla\rho_{h,\Delta t}, U_{h,\Delta t}, v_{h,\Delta t}\right) - \frac{1}{2}\int_0^T \left\langle U_{h,\Delta t}\frac{\partial}{\partial t}\breve{\rho}_{h,\Delta t}, v_{h,\Delta t} \right\rangle \\ &= \int_0^T \left\langle -\frac{c}{Re}(U_{h,\Delta t}.\nabla\rho_{h,\Delta t}) + \frac{1}{2Fr^2}(\rho_{h,\Delta t})^2, \nabla v_{h,\Delta t} \right\rangle \end{aligned} \tag{95}$$

Using the convergence results obtained earlier, we can pass to the limit in the variational formulation (95) of the discrete velocity equation $(53)_3$ to obtain:

$$\begin{aligned} &-\int_0^T \langle \rho U, \partial_t v\rangle - \langle \rho_0 U_0, v_0\rangle + \int_0^T c\left(\rho U - \frac{c}{Re}\nabla\rho, U, v\right) \\ &+ \int_0^T a\left(\rho, U, v\right) \\ &= \int_0^T \left\langle -\frac{c}{Re}(U.\nabla\rho) + \frac{1}{2Fr^2}\rho^2, \nabla v\right\rangle + \frac{1}{2}\int_0^T \left\langle U\frac{\partial}{\partial t}\rho, v\right\rangle \end{aligned} \tag{96}$$

Therefore, considering the variational formulation given in (12), we can deduce that this equation is equivalent to the following equation where we replace the test function v with a test function $\phi \in \mathcal{D}(Q_T)$ with $Q_T = [0,T] \times Q$.

$$-\int_{Q_T}\rho U\partial_t\phi dxdydt - \int_{Q_T}\Big(\rho U\otimes U - \frac{1}{Re}\nabla U$$
$$-\frac{c}{Re}(\nabla\rho\otimes U + U\otimes\nabla\rho)\Big).\nabla\phi dxdydt$$
$$-\frac{1}{2Fr^2}\int_{Q_T}\rho^2\,\mathrm{div}(\phi)dxdydt$$
$$+\frac{c}{Re}\int_{Q_T}\mathrm{div}(U)\nabla\rho.\phi dxdydt$$
$$=\int_Q \rho_0 U_0\phi(0,x,y)dxdy \tag{97}$$

Which is equivalent to the equation (11).Then, the limit function (ρ, U) satisfies the weak formulation (11) of the Kazhikhov-Smagulov model in the distribution sense on $(0;T)$. Consequently, we conclude the proof of Theorem 4.1.

5. Numerical Results

This section introduces two types of numerical simulations to demonstrate that the scheme provides accurate approximations of density and velocity for variable-density fluids, exemplified by the Kazhikhov-Smagulov equations. First, we compare the numerical solution with an analytical solution from the literature. Then, we examine the evolution of L^∞, L^2, and L^1 errors with respect to Δt and $(\Delta t)^2$.

To do this, we consider the following system as the analytical solution.

$$\begin{cases}\rho_{ex} = 2+\sin(y)\cos(x)\sin(t),\\ U_{ex} = -4\begin{pmatrix} y(x-1)^2(x+1)^2(y+1)(y-1)\\ -x(y-1)^2(y+1)^2(x+1)(x-1)\end{pmatrix}\end{cases} \tag{98}$$

Here we take $Q=[-1;1]^2$, $\mathcal{W}_h=\mathbb{P}_1$ and $\mathcal{V}_h=\mathbb{P}_2$.

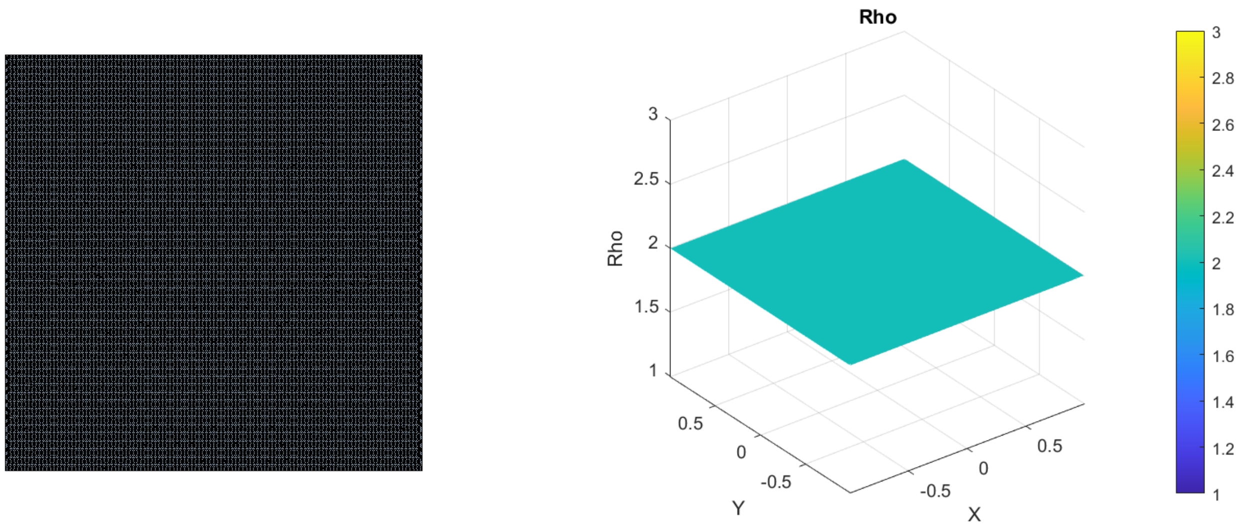

Fig. 2. At left Meshing of domain $Q=[-1;1]^2$ and at right the exact density for $h=0.0301$ and $n=91772$ triangles.

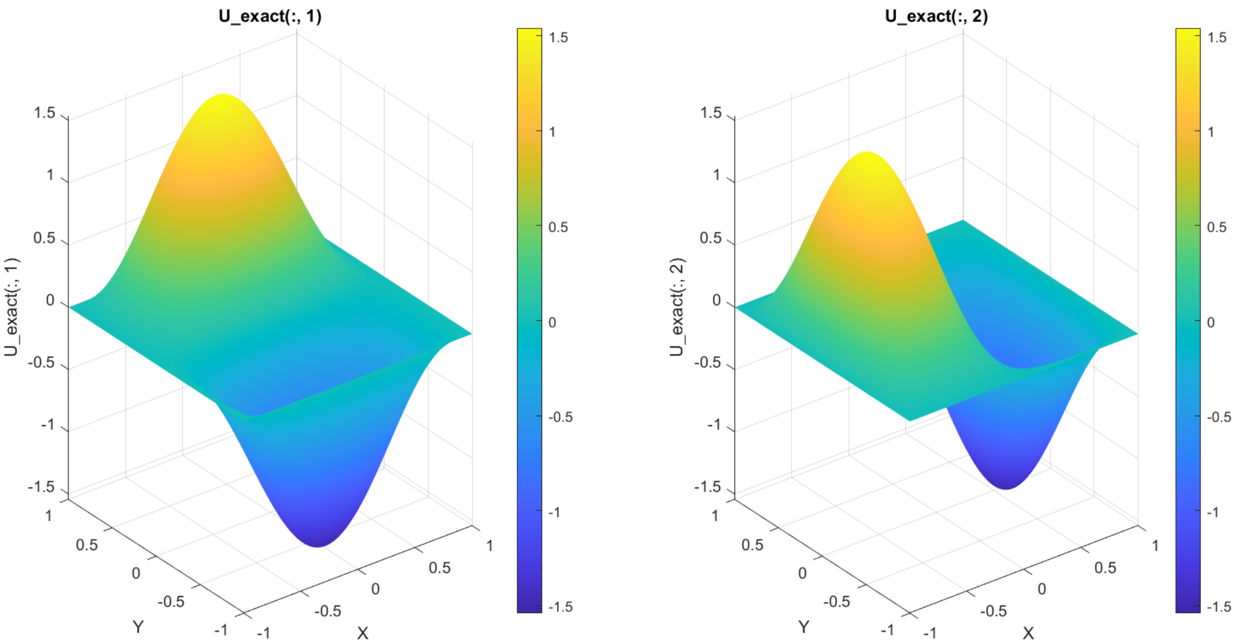

Fig. 3. The two components of exact velocity.

Table 1
Errors in norms as a function of h and $\Delta t = 5.10^{-2}$.

n (triangles)	29	45	47	88	137
h	1.7666	1.3938	1.3552	0.9880	0.7882
$err\text{-}\rho\text{-}L^1$	1.35×10^{-7}	1.40×10^{-7}	1.39×10^{-7}	1.45×10^{-7}	1.48×10^{-7}
$err\text{-}\rho\text{-}L^2$	7.03×10^{-8}	7.17×10^{-8}	7.16×10^{-8}	7.37×10^{-8}	7.51×10^{-8}
$err\text{-}\rho\text{-}L^\infty$	4×10^{-8}	4×10^{-8}	4×10^{-8}	4×10^{-8}	4×10^{-8}
$err\text{-}U_1\text{-}L^1$	2.83×10^{-2}	5.73×10^{-2}	3.17×10^{-2}	1.05×10^{-2}	4.66×10^{-3}
$err\text{-}U_1\text{-}L^2$	3.62×10^{-2}	1.69×10^{-1}	3.40×10^{-2}	1.09×10^{-2}	4.75×10^{-3}
$err\text{-}U_1\text{-}L^\infty$	5.80×10^{-2}	7.01×10^{-1}	4.38×10^{-2}	1.38×10^{-2}	5.40×10^{-3}
$err\text{-}U_2\text{-}L^1$	4.16×10^{-2}	6.28×10^{-2}	4.51×10^{-2}	1.31×10^{-2}	6.15×10^{-3}
$err\text{-}U_2\text{-}L^2$	5.59×10^{-2}	1.51×10^{-1}	6.83×10^{-2}	1.55×10^{-2}	7.53×10^{-3}
$err\text{-}U_2\text{-}L^\infty$	1.03×10^{-1}	5.95×10^{-1}	1.62×10^{-1}	3.43×10^{-2}	1.64×10^{-2}

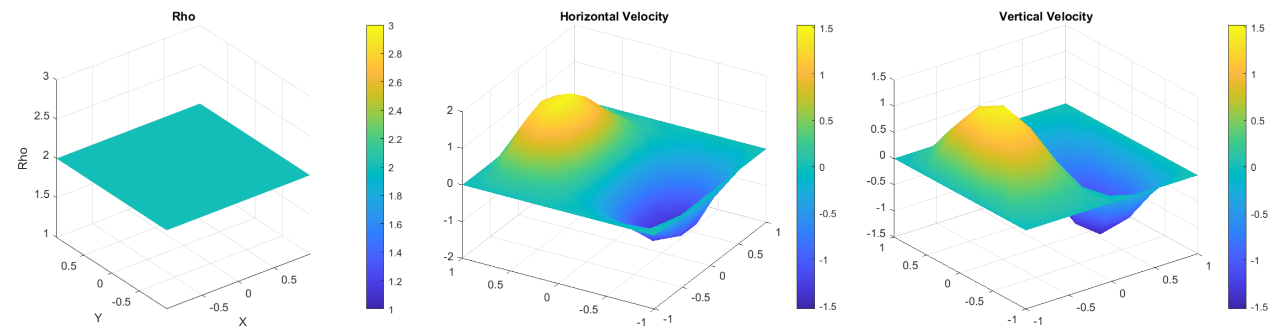

Fig. 4. Approximate solution by the hybrid finite volume-finite element method (Density, Horizontal velocity, Vertical velocity) for $\Delta t = 5 \cdot 10^{-2}, c = 10^{-3}$ and $Re = 5000$ with $h = 0.7882$.

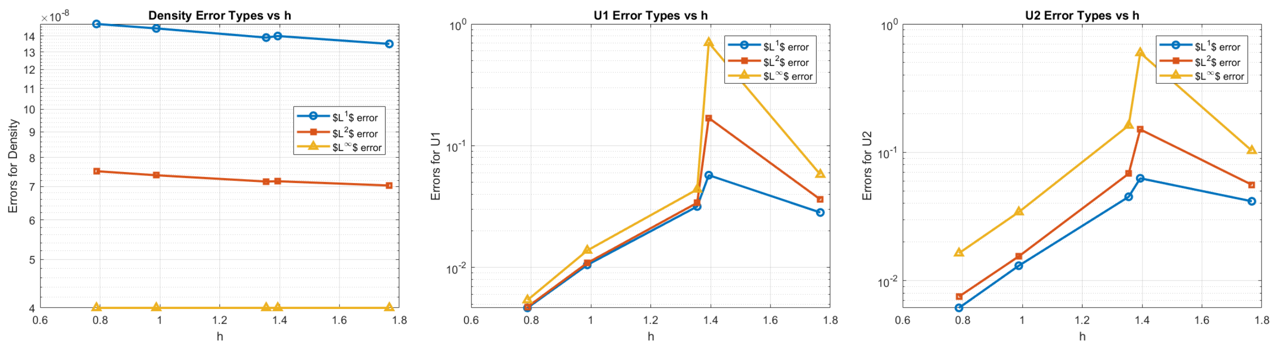

Fig. 5. Evolution of errors in norms L^1, L^2, and L^∞ as a function of h for the density, the horizontal velocity, and the vertical velocity respectively.

Interpretation of the results:

(i) We conducted numerical simulations to compare the exact and numerical solutions for both the density and the two components of velocity. The simulations were performed over a range of spatial discretization values, h, varying from 1.7666 to 0.7882. This range enables us to observe the behavior of the solutions under different resolutions, thereby highlighting the performance of the hybrid method.

(ii) The results of the numerical simulations reveal significant findings regarding the convergence of the solutions:

- The errors measured in norms L^1 and L^2 for the three different solution types consistently approach zero as the spatial discretization parameter h tends toward zero. This indicates that our numerical method is converging towards the exact solutions with increasing resolution.
- Additionally, the L^∞ norm of the two velocity components also converges towards zero. This is a positive indicator of the method's

accuracy, ensuring that the computed velocities are reliably approximating the true velocities.

- However, it is noteworthy that the error in the L^∞ norm for the density remains stable at 4×10^{-8} throughout the simulations. This behavior suggests that while the numerical method performs well in achieving convergence for the velocity components, the accuracy for the density may require further examination or refinement of the method.

(iii) Several figures were generated to illustrate the evolution of errors as a function of the spatial discretization step h. These visual representations provide insightful clarity into how the errors decrease with smaller values of h, emphasizing the effectiveness of the hybrid method across different solution types.

6. Conclusion

The application of the hybrid finite volume - finite element method to Kazhikhov-Smagulov type equations has proven to be effective in producing convergent solutions for both the density and the velocity fields. While the method shows promising results, particularly in the convergence of velocity errors, additional investigation may be warranted to enhance the accuracy of the density representation. The graphical analysis further supports these conclusions by illustrating the relationship between discretization and error reduction.

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