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Bispecial and separator factors in the generalized *Thue-Morse* word

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***Abstract*:**

In this paper, we study some combinatorics properties of the generalized Thue-Morse word $\mathbf{t}_q$ generated by the morphism: $\mu_q(k) = k(k+1)(k+2)...(k+q-1)$, where $k \in \{0, 1, ..., q\text{-}1\}$, $q \geq 2$ and the letters k are expressed modulo q. We focus our study on the bispecial factors in these sequences. More precisely, we describe bispecial biprolongable and bispecial q-prolongable factors and we show that they are the only bispecial factors in $\mathbf{t}_q$. We end by the complete description of the separator factors.

***Keywords*:** Infinite word; Biprolongable bispecial factors; q-prolongable bispecial factors; Separator factors.

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1 Introduction

Word Combinatorics is a branch of mathematics and computer science that applies combinatorial analysis to finite or infinite words. This branch is developed from several branches of mathematics, such as numbers theory, group theory, probability and combinatorics. It has links with computer themes, such as text algorithms, research of patterns, compression of texts, etc. The purpose of word combinatorics is to study finite or infinite words properties. It goes back to the works of Axel Thue on sequences without repetitions of symbols, at the beginning of the 20th century.

Thue-Morse word $\mathbf{t}_2$ on the binary alphabet $\mathcal{A}_2 = \{0,1\}$ is the infinite word $\mathbf{t}_2$, beginning by 0 and generated by the morphism μ_2 defined by: $\mu_2(0) = 01$ and $\mu_2(1) = 10$.

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Example 1.1. *Construction of* $\mathbf{t}_2$

$$\mu_2(0) = 01$$
$$\mu_2^2(0) = \mu_2(\mu_2(0)) = 0110$$
$$\mu_2^3(0) = \mu_2(\mu_2^2(0)) = 01101001$$
$$\vdots$$

The image of a factor by μ_2 *is entirely determined by the images of the letters that constitute it. Thus, the first letters of* $\mathbf{t}_2$ *are given by:*

$$\mathbf{t}_2 = (\mu_2(0))^\infty = 0110100110010110100101\,100110100110\ldots.$$

Thue-Morse word was used for the first time implicitly by Eugene Prouhet in 1851, to give a solution to a problem in number theory, since called the Prouhet-Tarry-Escott Problem [1]. In 1921 Morse used the word $\mathbf{t}_2$ to give an example of a non-periodic recurrent sequence, thus solving a differential geometry problem [2]. The word $\mathbf{t}_2$ has been extensively studied and is among the most studied classical infinite words [3–7]. It can be generalized over an alphabet of size q into the infinite word $\mathbf{t}_{b,q}$ generated by the morphisms $\mu_{b,q}$ defined by:

$$\mu_{b,q}(k) = k(k+1)\ldots(k+b-1)$$

where the letters k are expressed modulo q and $2 \leq b \leq q$. In this paper, we study some combinatorial properties of the word $\mathbf{t}_q(b = q)$ generated by the morphism μ_q defined by:

$$\mu_q(k) = k(k+1)\ldots(k+q-1),$$

where the letters k are expressed modulo q. In particular, for $q = 2$ we obtain the Thue-Morse word $\mathbf{t}_2$. A separator factor in $\mathbf{t}_q$ is a factor which separates two consecutive squares of letters in $\mathbf{t}_q$. In [8], the authors use the separator factors for the study of return words of ternary Thue-Morse word. The bispecial and separator factors have a great interest in the study of singular factors and recurrence functions of infinite words [9–11]. The special factors are also used in the determination of the complexity functions of the infinite words [12]. The paper is structured as follows. After some few definitions and notations in the Section 2, we present a development on the bispecial biprolongable and q-prolongable factors of $\mathbf{t}_q$ in the Section 3. Then, we show that the only bispecial factors of the word $\mathbf{t}_q$ are biprolongable and the q-prolongable. We end by the study of separator factors in the Section 4.

2 Definitions and notations

An alphabet $\mathcal{A}$ is a set of finite symbols. The set of finite words over $\mathcal{A}$ is denoted by $\mathcal{A}^*$.

Definition 2.1. An infinite word is an infinite sequence of letters on $\mathcal{A}$. $\mathcal{A}^\omega$ is the set of all infinite word over the alphabet $\mathcal{A}$ and the empty word is the word which does not contain any letter.
The empty word is denoted by ε.

Definition 2.2. A finite word w is a factor of u (finite or infinite) if w appears in u that means that there exists some words $u' \in \mathcal{A}^*$, $u'' \in \mathcal{A}^\omega \cup \mathcal{A}^*$ such that $u = u'wu''$. If $u' = \varepsilon$ (resp. $u'' = \varepsilon$), then w is a prefix (resp. suffix) of u.

Let $u, v \in \mathcal{A}^*$; $n, m \in \mathbb{N}^*$ and $a_i, b_j \in \mathcal{A}$ $(1 \leq i \leq m, 1 \leq j \leq n)$. The word $u = a_1a_2 \ldots a_m$ and $v = b_1b_2 \ldots b_n$ are equal if $n = m$ and $a_i = b_i$ for all $i \in [\![1, n]\!]$.

The set of the factors of u is called language of u and denoted $F(u)$.

Definition 2.3. Let w be a factor of an infinite word u and $a, b \in \mathcal{A}$. If wa (resp. bw) is a factor of u, then a (resp. b) is called right (resp. left) extension of w. w is right (resp. left) prolongable if it has at least one right (resp. left) extension in u. w is right (resp. left) q-prolongable if wa (resp. wb) is a factor of u for all $a, b \in \mathcal{A}_q$ where $\mathcal{A}_q = \{0, 1, ..., q-1\}$. w is right (resp. left) special if it has at least two right (resp. left) extensions in u. w is called bispecial if it is both left special and right special.

Example 2.1. *Consider the ternary Thue-Morse word:*

$$\begin{aligned}\mathbf{t}_3 \;=\; & 012120201120201012201012120120\\ & 20101220101212001212020120 10\\ & 12120012120201120\ldots\end{aligned}$$

012 *is a bispecial triprolongable factor of* $\mathbf{t}_3$ *because* $00121, 20121, 10121, 10122, 10120, 20120$ *are factors of* $\mathbf{t}_3$.
0120 *is a bispecial biprolongable factor of* $\mathbf{t}_3$ *because* $10121, 20120, 20120$ *are factors of* $\mathbf{t}_3$.
0121 *and* 1202 *are left special factors of* $\mathbf{t}_3$ *because* $10121, 00121, 10121, 21202, 11202$ *and* 21202 *are factors of* $\mathbf{t}_3$.
1012 *is a right special factor of* $\mathbf{t}_3$ *because* $10121, 10120$ *and* 10122 *are factors of* $\mathbf{t}_3$.

Definition 2.4. An infinite word u is recurrent if any factor of u appears an infinitely times in u. It said to be uniformly recurrent if for all integer n it exists an integer n_0 such that any factor u of length n appears in any factor of u of length n_0.

Definition 2.5. A mapping Φ defined on monoid $\mathcal{A}^*$ is a morphism if for any $u, v \in \mathcal{A}^*$ one has: $\Phi(uv) = \Phi(u)\Phi(v)$.

Definition 2.6. Let $u \in \mathcal{A}^*$. The length of u is the number of letter which contains the word u. $|u|$ denotes the length of u.

Example 2.2. $|1202010122| = 10$.

Definition 2.7. Over $\mathcal{A}_q = \{0, 1, ..., q-1\}$, the mapping E_q is defined by:

$$E_q(i) = \begin{cases} i+1 \mod q \text{ if } i \in \{0, 1, \ldots, q-2\} \\ 0 \text{ if } i = q-1. \end{cases}$$

Definition 2.8. A morphism $\Phi : A^* \to B^*$ is non-erasing for concatenation if $\Phi(a) \neq \varepsilon$ for all $a \in \mathcal{A}$.

Definition 2.9. A substitution is a non-erasing morphism for the concatenation over the free monoid $\mathcal{A}^*$.

Definition 2.10. We call fixed point of a substitution σ any point of σ verifying $\sigma(u) = u$.

Example 2.3. *The Thue-Morse word* $\mathbf{t}_3$ *is the fixed point generated by the substitution* μ_3 *defined over the alphabet* $\mathcal{A}_3 = \{0, 1, 2\}$ *by* $\mu_3(0) = 012$, $\mu_3(1) = 120$ *and* $\mu_3(2) = 201$.

$$\begin{aligned}\mathbf{t}_3 &= \lim_{k\to+\infty} \mu_3^k(0)\\ &= 012120201120201012201012120120 20\\ &\quad 101220101212001212020120101212 00\\ &\quad 12120201120\ldots\end{aligned}$$

Definition 2.11. Let $\mathcal{A} = \{a_1, a_2, ..., a_d\}$ be an alphabet and φ be a substitution on A^*. Then φ is said to be marked on the left (resp. on the right), if the first (resp. last) letters of $\varphi(a_i)$ and $\varphi(a_j)$ differ, for any $i \neq j$. It is said to be marked if it is both marked on the right and on the left.

Definition 2.12. [13]. Let w be a factor of a fixed point u of a substitution φ. The word $v_0v_1v_2\ldots v_n \in F_{n+1}(u)$ is said to be an ancestor of w if

i) w is a factor of $\varphi(v_0v_1v_2\ldots v_n)$,

ii) w is neither a factor of $\varphi(v_1v_2\ldots v_n)$ nor $\varphi(v_0v_1v_2\ldots v_{n-1})$.

Let us designate by $Anc(w)$, the set of the ancestors of the factor w.

Example 2.4. *Let us consider the Thue-Morse word:*

$$\begin{aligned}\mathbf{t}_3 \;=\; & 0121202011202010122010121201202\\ & 10122010121200121202012010121200\\ & 12120201120\ldots\end{aligned}$$

The set of ancestors of 01 *in* $\mathbf{t}_3$ *is given by* $Anc(w) = \{0, 2, 11\}$.

Definition 2.13. Let w be a factor of an infinite word u. If w starts with a and ends with b then $a^{-1}w$ (resp. wb^{-1}) is the word obtained from w by erasing a (resp. b).

Definition 2.14. Let u be an infinite word over an alphabet $\mathcal{A}$. A square of letter in u is the factor of u of the form ii where $i \in \mathcal{A}$.

Example 2.5. *The squares of letter in* $\mathbf{t}_3$ *are* 00, 11 *and* 22.

Definition 2.15. Let ij be a factor of $\mathbf{t}_q$. Then ij is called border of $\mathbf{t}_q$ if $j \neq i + 1$.

Example 2.6. *For* $q = 3$, *the borders in* $\mathbf{t}_3$ *are* $00, 02, 11, 10, 22, 21$.

3 Bispecial factors of $\mathbf{t}_q$

3.1 Bispecial q-prolongable factors in $\mathbf{t}_q$

Lemma 3.1. *[14] Let u be a factor of $\mathbf{t}_q$. Then, there exists factors v, δ_1 and δ_2 of $\mathbf{t}_q$ such that $u = \delta_1\mu_q(v)\delta_2$ with $|\delta_1|, |\delta_2| \leq q - 1$. The decomposition is unique if $|u| \geq 2q$.*

Proposition 3.1. *Let u be a factor of $\mathbf{t}_q$. Then, u is right (resp. left) q-prolongable factor of $\mathbf{t}_q$ if and only if $\mu_q(u)$ is right (resp. left) q-prolongable factor of $\mathbf{t}_q$.*

Proof 3.1. *Let u be a right q-prolongable factor of $\mathbf{t}_q$. Then, for any $i \in \mathcal{A}_q$, ui is in $\mathbf{t}_q$. As a result, $\mu_q(u)i$ is in $\mathbf{t}_q$, since $\mu_q(i)$ starts with i.*
Conversely, let u be a factor of $\mathbf{t}_q$ such that $\mu_q(u)$ is right q-prolongable with $|u| \geq 2$. Then, $\mu_q(u)i$ is in $\mathbf{t}_q$, for all $i \in \mathcal{A}_q$. Thus, the factor $\mu_q(u)i$ ends with the first letter of the image of $\mu_q(i)$, for all $i \in \mathcal{A}_q$. Since $|\mu_q(u)i| \geq 2q + 1$, from the uniqueness in Lemma 3.1, it follows that the words $\mu_q(u)012...$, $\mu_q(u)123...$,..., $\mu_q(u)(q-2)(q-1)0...$ and $\mu_q(u)(q-1)01...$ are in $\mathbf{t}_q$. These q factors are written respectively $\mu_q(u0)$, $\mu_q(u1)$,..., $\mu_q(u(q-2))$, $\mu_q(u(q-1))$. Thus u is right q-prolongable in $\mathbf{t}_q$.

Now, let us show that each letter of $\mathcal{A}_q$ is right q-prolongable. Let $i \in \mathcal{A}_q$ such that $ij \in F(\mathbf{t}_q)$ with $j \in \mathcal{A}_q$, j fixed. Observe that ij comes from the image of $(i+1)j$ by μ_q. At the same, $(i+1)j$ comes from the image of $(i+2)j$ by μ_q. Successively we show that ij is in $\mathbf{t}_q$ for all letter i. That means that j is left q-prolongable. For all $j \in \mathcal{A}_q$ we show that j is right q-prolongable in $\mathbf{t}_q$. Thus, for $|u| \leq 1$, u is q-prolongable if $\mu_q(u)$ is q-prolongable. The proof is sim-

ilar for a factor u of $\mathbf{t}_q$ left q-prolongable.

Remark 3.1. *Let w be a factor of $\mathbf{t}_q$. Then, one has:*
1) w is bispecial factor if and only if $\mu_q(w)$ is bispecial factor.
2) w is left (resp. right) special factor if and only if $\mu_q(w)$ is left (resp. right) special factor.

In the following, we designate by $BSQ(\mathbf{t}_q)$, the set of the factors of $\mathbf{t}_q$ both right q-prolongable and left q-prolongable.
As a result of the Proposition 3.1, a factor u is in $BSQ(\mathbf{t}_q)$ if and only if $\mu_q(u)$ is in $BSQ(\mathbf{t}_q)$.

Proposition 3.2. *Let u be an element of $BSQ(\mathbf{t}_q)$. If $|u| \geq q$, there exists u' in $BSQ(\mathbf{t}_q)$ such that $u = \mu_q(u')$.*

Proof 3.2. *Let $u \in BSQ(\mathbf{t}_q)$ such that $|u| \geq q$. Then, the factor u is written in a unique way in the form $u = \delta_1\mu_q(u')\delta_2$, where u', δ_1 and δ_2 are factors of $\mathbf{t}_q$ with $|\delta_1|, |\delta_2| \leq q-1$. Let us verify that the factors δ_1 and δ_2 are empty. Assume that δ_1 and δ_2 are not empty. Since u is right q-prolongable, the factors $\delta_2 0$, $\delta_2 1$,..., $\delta_2(q-1)$ are in $\mathbf{t}_q$. Thus, one of the $\delta_2 i$ contains the square of some letter. That is impossible because the image of any letter does not contain a square. In the same way, we show that δ_1 is empty. Thus $u = \mu_q(u')$. According to Proposition 3.1, u' is in $BSQ(\mathbf{t}_q)$.*

Let us notice that the short elements of $BSQ(\mathbf{t}_q)$ are the bispecial q-prolongable factors of $\mathbf{t}_q$ of length strictly inferior than q. We denote by $BSQ_m(\mathbf{t}_q)$ the set of short elements of $BSQ(\mathbf{t}_q)$.

Proposition 3.3. *The set $BSQ_m(\mathbf{t}_q)$ is given by: $BSQ_m(\mathbf{t}_q) = \{i(i+1)\dots(i+k-2) : i \in \mathcal{A}_q, 2 \leq k \leq q\}$.*

Proof 3.3. *First, let us show that every letter of $\mathcal{A}_q$ is both right and left q-prolongable. This is equivalent to show that for all $i, j \in \mathcal{A}_q$, there exists a natural number n such that ij appears in $\mu_q^n((j+q-1)j)$. Observe that ij comes from the image of $(i+1)j$ by μ_q. Likewise, $(i+1)j$ comes from the image of $(i+2)j$ by μ_q. Thus, ij comes from $\mu_q^2((i+2)j)$. Successively, we show that for any natural integer k, ij appears in $\mu_q^k((i+k)j)$. In particular, for $k = j+q-i-1$, ij appears in $\mu_q^{j+q-i-1}((j+q-1)j)$. As $(j+q-1)j$ is factor of $\mathbf{t}_q$, then ij is in $\mathbf{t}_q$. As a result, all the words of length 2 are in $\mathbf{t}_q$. Consequently, all the letters of $\mathcal{A}_q$ are both right q-prolongable and left q-prolongable.*
Let us show that for any letter i, $i(i+1)$ is both right q-prolongable and left q-prolongable. Observe that $i(i+1)$ is the prefix (resp. suffix) of $\mu_q(i)$ (resp. $\mu_q(i+2)$). As all letter of $\mathcal{A}_q$ is both right q-prolongable and left q-prolongable, then $\mu_q(i)$ and $\mu_q(i+2)$ are bispecial q-prolongable . As a result, $i(i+1)$ is bispecial q-prolongable according to Proposition 3.1.
Similarly to the previous, we show successively that $i(i+1)(i+2), i(i+1)(i+2)(i+3), ..., i(i+1)(i+2)...(i+q-2)$ are both right q-prolongable and left q-prolongable factors.

Theorem 3.1. *The set $BSQ(\mathbf{t}_q)$ is given by:*

$$BSQ(\mathbf{t}_q) = \bigcup_{n\geq 0}\{\mu_q^n(i(i+1)\dots(i+k-2)) : i \in \mathcal{A}_q, 2 \leq k \leq q\} \cup \{\varepsilon\}.$$

Proof 3.4. *Let u be an element of $BSQ(\mathbf{t}_q)$ of length greater or equal to q. According to Proposition 3.2, there exists u' in $BSQ(\mathbf{t}_q)$ such that $u = \mu_q(u')$. Thus, to obtain the* *set $BSQ(\mathbf{t}_q)$, it suffices to find the elements of $BSQ_m(\mathbf{t}_q)$ since the others are obtained by successive applications of μ_q. These factors are $i, i(i+1), ..., i(i+1)(i+2)...(i+q-3)$ and $i(i+1)(i+2)...(i+q-2)$.*

Corollary 3.1. *Let u be a factor of $\mathbf{t}_q$ right q-prolongable. If $|u| = q^k$ or $|u| = l \times q^k$, $k \geq 0$, $2 \leq l \leq q-1$, then u is left q-prolongable.*

Proof 3.5. *Let u be a factor of $\mathbf{t}_q$ right q-prolongable such that $|u| = lq^k$, $k \geq 0$ and $1 \leq l \leq q-1$. Then, one has:*

- *If $k = 0, |u| = l$. Consequently, u is either proper prefix or proper suffix of the image of a letter. Then u is left q-prolongable.*
- *If $k \geq 1$, $|u| = l \times q^k$ with $1 \leq l \leq q-1$. Then u decomposes into the form $\delta_1\mu_q(v)\delta_2$ where $v, \delta_1, \delta_2 \in F(\mathbf{t}_q)$. The factor u being right q-prolongable, δ_2 is the empty word. As a result $u = \delta_1\mu_q(v)$. Thus, one has:*

$$\begin{aligned} u = \delta_1\mu_q(v) &\implies |u| = |\delta_1| + |\mu_q(v)| \\ &\implies l \times q^k = |\delta_1| + q|v| \\ &\implies |\delta_1| = q(l \times q^{k-1} - |v|). \end{aligned}$$

It follows that $|\delta_1|$ is a multiple of q of length strictly less than q. As a result, $|\delta_1| = 0$. Which implies that $\delta_1 = \varepsilon$ and $u = \mu_q(v)$. Thus, $u = \mu_q(v)$ where v is a right q-prolongable factor of length q^{k-1}. By the same process, the factor v can be written $v = \mu_q(v')$, where v' is a right q-prolongable factor of length q^{k-2}. Successively, we get $u = \mu_q^k(i)$, $i \in \mathcal{A}_q$. By using Theorem 3.1 we conclude that u is left q-prolongable.

Proposition 3.4. *Let u be a right q-prolongable factor of $\mathbf{t}_q$. If u is left special, then it is left q-prolongable.*

Proof 3.6. *Let u be a right q-prolongable factor of $\mathbf{t}_q$ and left special. Then, u synchronizes in the form $\delta_1\mu_q(v_1)\delta_2$. As the factor u is right q-prolongable then $\delta_2 i$ is a factor of $\mathbf{t}_q$ for all $i \in \mathcal{A}_q$. That means δ_2 contains a square of letter. Which is absurd because no letter image prefix contains a square. So δ_2 is the empty word. Moreover, as u is left special, then δ_1 is empty; otherwise, δ_1 would be proper suffix of the image of a letter and therefore u would be extended uniquely to the left. Thus, u synchronizes as $u = \mu_q(v_1)$, where v_1 is a factor of $\mathbf{t}_q$. The morphism μ_q being marked to the left, then v_1 is left special. Moreover, u is right q-prolongable according to Proposition* 3.1. *So, v_1 synchronizes as $v_1 = \mu_q(v_2)$, $v_2 \in F(\mathbf{t}_q)$. Successively, we end up to $u = \mu_q^k(v_k)$ with $k \geq 0$ and v_k, a right q-prolongable factor, left special and of length inferior or equal to $q-1$. Consequently, v_k is left q-prolongable according to Theorem* 3.1.

Proposition 3.5. *Let $u \in BSQ(\mathbf{t}_q)$. Then, there exists a unique letter i in $\mathcal{A}_q$ such that iu (resp. ui) is right (resp. left) q-prolongable.*

Proof 3.7. *Let $u \in BSQ(\mathbf{t}_q)$, $i \in \mathcal{A}_q$ and a be the first letter of u. Suppose that iu is right q-prolongable. We have the following two cases.*

***Case 1**: $|iu| \leq q-1$. Then we have $iu \in BSQ(\mathbf{t}_q)$ according to Corollary* 3.1. *Consequently, $i = a-1$, we have any element of $BSQ(\mathbf{t}_q)$ of length inferior than $q-1$ is either prefix or suffix of the image of a letter.*

***Case 2**: $|iu| \geq q+1$. Then, according to Proposition* 3.1 *it exists an element u_1 of $BSQ(\mathbf{t}_q)$ such that $iu = i\mu_q(u_1)$. Note that i is the last letter of $\mu_q(i-q+1)$. As*

iu is right q-prolongable, $(i-q+1)u_1$ is also right q-prolongable by Proposition 3.1. *Likewise, if $|(i-q+1)u_1| \geq q+1$ then, it exists an element u_2 of $BSQ(\mathbf{t}_q)$ such that $(i-q+1)u_1 = (i-q+1)\mu_q(u_2)$ according to Proposition* 3.1. *Observe that $i-q+1$ is the last letter of $\mu_q(i-2q+2) = \mu_q(i+2)$, It follows that $(i+2)u_2$ is right q-prolongable. By proceeding successively, we show that $iu = i\mu_q^n(u_n)$ where $u_n \in BSQ(\mathbf{t}_q)$ of length less than $q-1$. Moreover, i is the last letter of $\mu_q^n(i-n(q-1))$. By Proposition* 3.1 *we show that $(i-n(q-1))u_n$ is right q-prolongable of length less than q. From Theorem* 3.1, *$(i-n(q-1))u_n$ is either a prefix or a suffix of the image of a letter. Consequently, $a = i-n(q-1)-1$. So $i = a+n(q-1)+1$. The uniqueness of i is due to the uniqueness of a, q and n.*
In the same way, one treats the case of the left q-prolongable factors.

Proposition 3.6. *For all positive integer n, $\mathbf{t}_q$ admits exactly q right (resp. left) q-prolongable factors of length n.*

Proof 3.8. *Note that $0, 1, \ldots, q-2$ and $q-1$ are the right q-prolongable factors of length* 1. *Let us show that every right q-prolongable factor of length n is a suffix of a unique right q-prolongable factor of length $n+1$.*
Let w be a right q-prolongable factor of length n. If w admits a unique left extension a, then aw is a right q-prolongable factor since $\mathbf{t}_q$ is recurrent. If it admits at least two left extensions, then w is in $BSQ(\mathbf{t}_q)$ by Proposition 3.4 *and one of these left extensions is right q-prolongable by Proposition* 3.5. *Thus, the number of right q-prolongable factors of length $n+1$ is equal to the number of right q-prolongable factors of length n of $\mathbf{t}_q$. The case of left q-prolongable factors is treated in a similar way.*

The following remark is a consequence of the Proposition 3.6.

Proposition 3.7. *Let u be a right (resp. left) q-prolongable factor of $\mathbf{t}_q$. If u is of length kq, $k \geq 1$, then there exists a factor v of $\mathbf{t}_q$, right (resp. left) q-prolongable such as $u = \mu_q(v)$.*

Proof 3.9.

1. *Assume that u is right q-prolongable factor of $\mathbf{t}_q$ of length kq, $k \geq 1$. Then u can be written as $u = \delta_1\mu_q(v)\delta_2$ with $|\delta_1|, |\delta_2| \leq q-1$. As u is right special then δ_2 is the empty word and $u = \delta_1\mu_q(v)$. Thus, one has:*

$$\begin{aligned} u = \delta_1\mu_q(v) &\implies |u| = |\delta_1\mu_q(v)| \\ &\implies |u| = |\delta_1| + |\mu_q(v)| \\ &\implies |\delta_1| = |u| - |\mu_q(v)| \\ &\implies |\delta_1| = kq - q|v| \\ &\implies |\delta_1| = q(k-|v|). \end{aligned}$$

   Consequently, $|\delta_1|$ is a multiple of q of length strictly less than q. Hence $|\delta_1| = 0$ and $u = \mu_q(v)$. Moreover, v is right q-prolongable because $\mu_q(v)$ is right q-prolongable.

2. *Assume that u is left q-prolongable factor of $\mathbf{t}_q$ of length kq, $k \geq 1$. we proceed as in 1 to show that $u = \mu_q(v)$.*

Corollary 3.2. *For all positive integer n, the right (resp. left) q-prolongable factors of length n begin (resp. end) with distinct letters .*

Proof 3.10. *We proceed by induction on n. We know that the letters are bispecial q-prolongable factors of length* 1.
Suppose that the right q-prolongable factors of $\mathbf{t}_q$ of length inferior or equal to n begin with the distinct letters. Let u_1 and u_2 be two factors of $\mathbf{t}_q$ left q-prolongable of length n. We distinguish the following cases.
***Case 1**: n is multiple of q. Then, there are factors v_1 and v_2 of $\mathbf{t}_q$ such that $u_1 = \mu_q(v_1)$ and $u_2 = \mu_q(v_2)$. Suppose there exists a letter a of $\mathcal{A}_q$ such that au_1 and au_2 are right q-prolongable. Without any lost of generality, let us take $a = 0$. Consequently, the factors $12\dots(q-1)0u_1$ and $12\dots(q-1)0u_2$ are left q-prolongable in $\mathbf{t}_q$. These factors can be written respectively $\mu_q(1v_1)$ and $\mu_q(1v_2)$. By Proposition* 3.1, *$1v_1$ and $1v_2$ are right q-prolongable factors. This contradicts the recurrence hypothesis since $1v_1$ and $1v_2$ are of length inferior than n.*
***Case 2**: $n-1$ is multiple of q. Then, there are factors v_1 and v_2 of $\mathbf{t}_q$ right q-prolongable such that $u_1 = i\mu_q(v_1)$ and $u_2 = j\mu_q(v_2)$, $i, j \in \mathcal{A}_q$. As i and j are suffix of images of letters, then i and j have each a unique left extension. Since they are distinct by hypothesis (u_1 and u_2 start with distinct letters), their extensions are distinct. For the cases where $n-k$ is multiple of q for $k \in \{2, ..., q-1\}$ we proceed similarly to case 2.*

3.2 Bispecial biprolongable factors in $\mathbf{t}_q$

Let us denote by $BSB(\mathbf{t}_q)$ the set of factors of $\mathbf{t}_q$ both right biprolongable and left biprolongable.

Proposition 3.8. *Let w be a factor of $\mathbf{t}_q$. Then w is right (resp. left) biprolongable if and only if $\mu_q(w)$ is right (resp. left) biprolongable.*

Proof 3.11. *Let w be a right biprolongable factor of $\mathbf{t}_q$. Then there exists $i, j \in \mathcal{A}_q$ with $i \neq j$ such that $wi, wj \in F(\mathbf{t}_q)$. So we have $\mu_q(w)i, \mu_q(w)j \in F(\mathbf{t}_q)$, since the image of i (resp. j) begins with i (resp. j). Thus, $\mu_q(w)$ is right biprolongable.*
Conversely, let w be a factor of $\mathbf{t}_q$ such that $\mu_q(w)$ is right biprolongable. Then there exists $i, j \in \mathcal{A}_q$ with $i \neq j$ such that $\mu_q(w)i, \mu_q(w)j \in F(\mathbf{t}_q)$. Observe that the factor $\mu_q(i)$ (resp. $\mu_q(j)$) begins with i (resp. j) . As a result, $\mu_q(w)\mu_q(i) = \mu_q(wi)$ (resp. $\mu_q(w)\mu_q(j) = \mu_q(wj)$) is a factor of $\mathbf{t}_q$, since $\mathbf{t}_q$ is recurrent. Thus, w is right biprolongable .
We proceed in the same way for the case of left biprolongable factors.

Proposition 3.9. *Let $w \in BSB(\mathbf{t}_q)$ verifying $|w| \geq 2q$. Then, there exists $v \in BSB(\mathbf{t}_q)$ such that $w = \mu_q(v)$.*

Proof 3.12. *Let $w \in BSB(\mathbf{t}_q)$ verifying $|w| \geq 2q$. Then w can be synchronized in a unique way as $w = \delta_1\mu_q(v)\delta_2$ with $|\delta_1|, |\delta_2| \leq q-1$. As w is right biprolongable, there exists $i, j \in \mathcal{A}_q$ with $i \neq j$ such that $\delta_2 i, \delta_2 j \in F(\mathbf{t}_q)$. δ_2 being a proper prefix of an image of letter this is only possible when $\delta_2 = \varepsilon$. We proceed in the same way to show that $\delta_1 = \varepsilon$. So, $w = \mu_q(v)$ and $v \in BSB(\mathbf{t}_q)$ according to Proposition* 3.8.

Define the morphism E_q on $\mathcal{A}_q^*$ by: $E_q(i) = i + 1 \mod q, \forall i \in \mathcal{A}_q$.

Remark 3.2. *Note that by the Definition of E_q, the following statements hold:*

(i) *The application E_q is bijective;*

(ii) $E_q \circ \mu_q = \mu_q \circ E_q$.

As E_q is bijective, we define E_q^{-1} as follows.

Definition 3.1. The reciprocal application E_q^{-1} of E_q is defined by:

$$E_q^{-1} : \mathcal{A}_q \rightarrow \mathcal{A}_q$$
$$i \mapsto \begin{cases} q-1 \text{ if } i = 0 \\ i-1 \mod q \text{ if} \\ i \in \{1, 2, \dots, q-1\}. \end{cases}$$

Proposition 3.10. *For all $i \in \mathcal{A}_q$, consider the sequence $(u_n^{(i)})_{n\geq 0}$ in $\mathcal{A}_q^*$ defined by:*

$$\begin{cases} u_0^{(i)} = i \\ u_{n+1}^{(i)} = \prod_{j=0}^{q-1} u_n^{(i+j)}, \forall n \geq 0. \end{cases}$$

Then, for all integer n, $u_n^{(i)} = \mu_q^n(i)$.

Proof 3.13. *We proceed by induction on n. We have $u_0^{(i)} = i = \mu_q^0(i)$.*
Let us assume that for $n \geq 1$, $u_n^{(i)} = \mu_q^n(i)$ and let us show that $u_{n+1}^{(i)} = \mu_q^{n+1}(i)$. We have:

$$\begin{aligned} u_{n+1}^{(i)} &= \prod_{j=0}^{q-1} u_n^{(i+j)} \\ &= \prod_{j=0}^{q-1} \mu_q^n(i+j) \\ &= \mu_q^n(i)\mu_q^n(i+1)\mu_q^n(i+2)\dots\mu_q^n(i+q-1) \\ &= \mu_q^n(i(i+1)(i+2)\dots(i+q-1)) \\ &= \mu_q^n(\mu_q(i)) \\ &= \mu_q^{n+1}(i). \end{aligned}$$

Thus, for all $n \geq 0$, $u_n^{(i)} = \mu_q^n(i)$.

Proposition 3.11. *The language of $\mathbf{t}_q$ is stable under the application E_q.*

Proof 3.14. *Let $w \in F_n(\mathbf{t}_q)$. Then by the uniform recurrence of $\mathbf{t}_q$ there exists $n_0 \in \mathbb{N}$ such that w is factor of $\mu_q^{n_0}(0)$. As result $E_q(w)$ appears in $E_q(u_{n_0}^{(0)}) = E_q(\mu_q^{n_0}(0)) = \mu_q^{n_0}(1)$. Thus, $E_q(w)$ is in $F(\mathbf{t}_q)$ because $\mu_q^{n_0}(1) \in F_n(\mathbf{t}_q)$. Thus we have $E_q(w) \in F_n(\mathbf{t}_q)$ because $\mu_q^{n_0}(1)$ appears in $\mathbf{t}_q$. So, $E_q(F_n(\mathbf{t}_q)) \subset F_n(\mathbf{t}_q)$. By bijectivity of E_q, it results that $E_q(F_n(\mathbf{t}_q)) = F_n(\mathbf{t}_q)$.*

Let w in $F(t_q)$. Then we denote by $Ext^+(w)$(resp. $Ext^-(w)$) the set of right(resp. left) extensions of w. We also denote by $BSB_m(\mathbf{t}_q)$ the set of short bispecial biprolongable factors of $\mathbf{t}_q$. Then, we have the following result.

Proposition 3.12. *The set $BSB_m(\mathbf{t}_q)$ is given by:*

$$BSB_m(\mathbf{t}_q) = \{\mu_q(i)i(i+1)\dots(i+k-2) : i \in \mathcal{A}_q,$$
$$2 \leq k \leq q\}.$$

Moreover, for all $i \in \mathcal{A}_q$ and for all $k \in [\![2, q]\!]$ we have:

$$Ext^+(\mu_q(i)i(i+1)\dots(i+k-2)) = \{E_q^{k-1}(i), E_q^k(i)\}$$

and

$$Ext^-(\mu_q(i)i(i+1)\dots(i+k-2)) = \{E_q^{q-1}(i), E_q^{q-2}(i)\}.$$

Proof 3.15. *By Proposition 3.11, it suffices to treat the case $i = 0$ since the others cases can be obtained by successive applications of E_q. So, consider the factor $\mu_q(0)01\dots(k-2)$. We distinguish the following cases.*
Case 1: *$k = 2$. Observe that the set of ancestors of $\mu_q(0)0$ is given by:*

$$Anc(\mu_q(0)0) = \{ii : i \in \mathcal{A}_q\}.$$

• *If $i = 1$, observe that $\mu_q(0)0$ is suffix $\mu_q(11)$. As 11 is right prolongable by 2 then one has:*

$$\mu_q(112) = 12\ldots\underbrace{012\ldots0}_{\mu_q(0)0}23\ldots01. \qquad (1)$$

Thus, $\mu_q(0)0$ is right prolongable by 2.

• *If $i \in \mathcal{A}_q\backslash\{1\}$, one notices that $\mu_q(0)0$ is not suffix of $\mu_q(ii)$ (the suffix $i+q-1$ of $\mu_q(ii)$ is different from 0 for all $i \in \mathcal{A}_q\backslash\{1\}$). Thus, one has:*

$$\mu_q(ii) = i(i+1)\ldots\ldots \underbrace{01\ldots(i+q-1)i(i+1)\ldots0}_{\mu_q(0)0}1\ldots(i+q-1) \qquad (2)$$

Hence $\mu_q(0)0$ is right prolongable by 1. From (1) and (2) it follows that $\mu_q(0)0$ is right biprolongable and their right extensions are $1 = E_q(0)$ and $2 = E_q^2(0)$.

Case 2: *$k \in [\![3, q]\!]$. Observe that the set of ancestors of $\mu_q(0)01\ldots(k-2)$ is given by:*

$$\begin{aligned} Anc(\mu_q(0)01\ldots(k-2)) &= \{ii : i \in \mathcal{A}_q\backslash\{1,2,\\ &\qquad \ldots,(k-2)\}\}\\ &= \{00,(k-1)(k-1),kk,\\ &\qquad \ldots,(q-1)(q-1)\}. \end{aligned}$$

As the factor $01\ldots(k-2)$ is a suffix of $\mu_q(k-1)$ then $\mu_q(0)01\ldots(k-2)$ is also suffix of $\mu_q((k-1)(k-1))$. Observe that the factor $(k-1)(k-1)$ is left prolongable by k. Thus, we have $(k-1)(k-1)k \in F(\mathbf{t}_q)$. So, it follows that $\mu_q((k-1)(k-1))$ is right prolongable by k. Therefore, $\mu_q(0)01\ldots(k-2)$ is right prolongable by k since it is suffix of $\mu_q((k-1)(k-1))$.

Moreover, for $ii \neq (k-1)(k-1)$, $\mu_q(0)01\ldots(k-2)$ is not suffix of $\mu_q(ii)$. By analogy to case 1 one checks that $\mu_q(0)01\ldots(k-2)$ is right prolongable by $k-1$. Consequently, $\mu_q(0)01\ldots(k-2)$ is right biprolongable and their right extensions are $k = E_q^k(0)$ and $k-1 = E_q^{k-1}(0)$. Let us show that $\mu_q(0)01\ldots(k-2)$ is left biprolongable. We distinguish two cases:

Case 1: *$k = 2$. We distinguish two possibilities.*

• *$i = 0$. Observe that $\mu_q(0)0$ is a prefix of $\mu_q(00)$. As 00 is left prolongable by $q-1$ then $(q-1)00 \in F(\mathbf{t}_q)$. So, by taking the image of this last factor one has:*

$$\mu_q((q-1)00) = (q-1)01\ldots(q-2)\underbrace{\mu_q(0)0}_{1\ldots(q-1)} \qquad (3)$$

Thus $\mu_q(0)0$ is left prolongable by $q-2$.

• *$i \in \mathcal{A}_q\backslash\{0\}$. We notice that $\mu_q(0)0$ is not prefix of $\mu_q(ii)$. Therefore, one has:*

$$\mu_q(ii) = i(i+1)\ldots(q-1)\underbrace{\mu_q(0)0}_{\ldots(i+q-1)} \qquad (4)$$

So $\mu_q(0)0$ is left prolongable by $q-1$. From (3) and (4) it follows that $\mu_q(0)(0)$ is left biprolongable and their left extensions are $q-1$ and $q-2$.

Case 2: *$k \in [\![3, q-1]\!]$. We proceed similarly as in the case 1 and we obtain that $\mu_q(0)01\ldots(k-2)$ is left biprolongable and their extensions are $q-1 = E_q^{q-1}(0)$ and $q-2 = E_q^{q-2}(0)$.*

Theorem 3.2. *The set $BSB(\mathbf{t}_q)$ of bispecial biprolongable factors is given by:*

$$BSB(t_q) = \bigcup_{n\geq 0}\{\mu_q^n(\mu_q(i)i(i+1)\ldots(i+k-2)) : i \in \mathcal{A}_q, 2 \leq k \leq q\}.$$

Proof 3.16. *According to Proposition 3.8,* *it suffices to consider the elements of $BSB(\mathbf{t}_q)$ of length at most $2q-1$ since the others can be obtained by successive application of μ_q. These factors are given by the set:*

$$BSB_m(\mathbf{t}_q) = \{\mu_q(i)i(i+1)\dots(i+k-2) : i \in \mathcal{A}_q, 2 \le k \le q\}.$$

Lemma 3.2. *Let w be a factor of $\mathbf{t}_q$ such that $|w| \le 2q-1$. If w contains a border then w is non-bispecial.*

Proof 3.17. *Assume that w contains a border. Then, there exists $(i,j) \in \mathcal{A}_q \times (\mathcal{A}_q \backslash \{E_q(i)\})$ such that $w = w_1 ij w_2$ where $w_1, w_2 \in F(\mathbf{t}_q)$. We distinguish two cases:*
***Case 1**: $2 \le |w| \le q-1$. Then $|w_1|, |w_2| \le q-3$. So, $w_1 i$ is suffix of $\mu_q(i+1)$ and jw_2 is prefix of $\mu_q(j)$. Thus, $w = w_1 ij w_2$ admits a unique right (resp. left) extension. Hence w is not a bispecial factor of $\mathbf{t}_q$.*
***Case 2**: $q \le |w| \le 2q-1$. Then w can not be written as an image of a letter since it contains a border. Thus, two cases arise:*
- *w does not contain an image of letter. Then w can be written $w = w_1 ij w_2$ where $w_1 i$ and jw_2 are respectively proper prefix and proper suffix of image of letters. Then w admits a unique left (resp. right) extension. So, w is not a bispecial factor of $\mathbf{t}_q$.*
- *w contains an image of letter. Then w can be written in the form $w = w_1'\mu_q(i+1)jw_2$ or $w = w_1 i\mu_q(j)w_2'$ where $|w_1'| < |w_1|$ and $|w_2'| < |w_2|$. So, w_1' and w_2' are respectively proper prefix and proper suffix of image of letters. Thus, w admits a unique left (resp. right) extension. Consequently, w is not bispecial factor of $\mathbf{t}_q$.*

Lemma 3.3. *Let $k \in [\![2, q]\!]$ and w be a factor of $\mathbf{t}_q$. Then one has:*
1) The factor $w = \mu_q(i)j(j+1)\dots(j+k-2)$ is left special if and only if $j = i+1$.
2) The factor $w = (j-k+2)(j-k+1)\dots(j-1)j\mu_q(i)$ is right special if and only if $j = E_q^{q-2}(i)$.

Proof 3.18. *1) $\Rightarrow$ Assume that w is left special and show that $j = i+1$.*
As w is left special then it is proper prefix of q-prolongable bispecial factor $\mu_q(ij)$. By the Proposition 3.1, ij is a q-prolongable bispecial factor. As $|ij| = 2$ then ij is a prefix of image of letter. Hence, $j = i+1$.
$\Leftarrow$ Assume that $j = i+1$. Then $i(i+1)$ is the unique ancestor of $w = \mu_q(i)j(j+1)\dots(j+k-2)$. As $i(i+1)$ is left special factor then $\mu_q(i(i+1))$ is left special. But w is a prefix of $\mu_q(i(i+1))$. Then w is also left special. Nevertheless, w is not right special because its suffix $j(j+1)\dots(j+k-2)$ is a proper prefix of $\mu_q(j)$.
2) We proceed similarly and we show that,

$$w = (j-k+2)(j-k+1)\dots(j-1)j\mu_q(i)$$

is right special if and only if $j = E_q^{q-2}(i)$ (it suffices to remark that $(i+q-1)i$ is the only ancestor of w).

Theorem 3.3. *The only bispecial factors of $\mathbf{t}_q$ are the bispecial biprolongable and bispecial q-prolongable factors.*

Proof 3.19. *Let w be a factor of $\mathbf{t}_q$ such that w is neither bispecial biprolongable nor bispecial q-prolongable. Show that w can not be a bispecial factor. We distinguish the following cases:*
***Case 1**: $2 \le |w| \le q-1$. Then, w can neither be prefix nor suffix of the image of a letter, otherwise it would be a q-prologable factor by Proposition 3.3. Therefore w contains a border. According to Lemma 3.2, w*

is not bispecial factor of $\mathbf{t}_q$.

Case 2: $|w| \in [\![q, 2q-1]\!]$. *As* w *is not a bispecial q-prolongable factor then* w *can not be written as image of letter. Thus, we have three possibilities.*

a) w *is neither left special nor right special. Then* w *is not a bispecial factor.*

b) w *is left special. Therefore,* w *can be written as* $w = \mu_q(i)j(j+1)\dots(j+k-2)$, $i, j \in \mathcal{A}_q$ *and* $2 \le k \le q$. *As* w *is left special then* $j = i+1 \neq i$ *by Lemma* 3.3. *So,*

$$\mu_q(i)j(j{+}1)\dots(j{+}k{-}2) = i(i{+}1)\dots\underbrace{(i+q-1)j}$$

$$(j+1)\dots(j+k-2)$$

and $(i+q-1)j$ *is a border. According to Lemma* 3.2, w *is not a bispecial factor.*

c) w *is right special. Therefore,* w *can be written,*

$$w = (j{-}k{+}2)(j{-}k{+}1)\dots(j{-}1)j\mu_q(i), i, j \in \mathcal{A}_q$$

$$\textit{and } 2 \le k \le q.$$

As w *is right special then* $j = E_q^{q-2}(i)$ *by the Lemma* 3.3. *So,* $i = j-q+2$ *and* ji *is a border since* $i = j-q+2 \neq j+1$. *According to Lemma* 3.2, w *is not a bispecial factor.*

Case 3: $|w| \ge 2q$. *Assume moreover* w *is bispecial.*

Then by Lemma 3.1, w *can be decomposed uniquely in the form* $w = \delta_1\mu_q(v_1)\delta_2$ *with* $\delta_1, \delta_2, v_1 \in F(\mathbf{t}_q)$ *and* $|\delta_1|, |\delta_2| \le q-1$. *As* w *is bispecial then* $\delta_1 = \delta_2 = \varepsilon$. *As a result,* $w = \mu_q(v_1)$. *Thus,* v_1 *is a bispecial factor by the Remark* 3.1. *We repeat the same process and show that* $v_1 = \mu_q(v_2)$. *Therefore,* $w = \mu_q^2(v_2)$. *Successively, we get* $w = \mu_q^k(v_k)$ *with* $2 \le |v_k| \le 2q-1$. *By hypothesis* w *is a bispecial factor which is neither biprolongable nor q-prolongable then* v_k *is also. Contradiction because according to the cases 1 and 2, any factor of* $\mathbf{t}_q$ *of length at most* $2q-1$ *which is neither biprolongable nor q-prolongable cannot be bispecial. So,* w *is not a bispecial factor.*

4 Separator factors of $\mathbf{t}_q$

In this section, we study the lengths and the different forms of separator factors of $\mathbf{t}_q$. Recall that a separator factor is a factor which separates two consecutives squares of letters in $\mathbf{t}_q$.

Remark 4.1. *For all* $i \in \mathcal{A}_q$, *one has:*

$$E_q^k(i) = i + k \mod q.$$

Proposition 4.1. *For all* $k \in [\![1, q-1]\!]$, $E_q^{(q-1)k}(i)$ *returns the last letter of* $\mu_q^k(i)$ *for all* $i \in \mathcal{A}_q$.

Proof 4.1. *Let us proceed by induction on* k.

For $k = 1$ *one has:* $\mu_q(i) = i(i+1)\dots(i+q-1)$ *and from Remark* 4.1 $i+q-1 = E_q^{q-1}(i)$.

By induction, assume that for $k \ge 2$, $E_q^{(q-1)k}(i)$ *is the last letter of* $\mu_q^k(i)$ *and check the proposition for the rank* $k{+}1$. *One has:*

$$\begin{aligned} \mu_q^{k+1}(i) &= \mu_q(\mu_q^k(i)) \\ &= \mu_q(i(i+1)\dots i+(q-1)k) \\ &= \mu_q(i)\mu_q(i+1)\dots\mu_q(i+(q-1)k). \end{aligned}$$

Thus, to find the last letter of $\mu_q^{k+1}(i)$ *it suffices to find the one of* $\mu_q(i{+}(q{-}1)k)$. *From the assumption of induction the last letter* *of $\mu_q(i+(q-1)k)$ is given by:*

$$\begin{aligned} E_q^{q-1}(i+(q-1)k) &= E_q^{q-1}(E_q^{(q-1)k}(i)) \\ &= E_q^{(q-1)+(q-1)k}(i) \\ &= E_q^{(q-1)(k+1)}(i). \end{aligned}$$

Remark 4.2. *For $k = q-1$, $E_q^{(q-1)^2}(i) = E_q(i)$.*

Indeed, we have:

$$\begin{aligned} Eq^{(q-1)^2}(i) &= i+(q-1)^2 \mod q \\ &= i+1+q(q-2) \mod q \\ &= i+1 \mod q \\ &= E_q(i). \end{aligned}$$

Lemma 4.1. *For all $i \in \mathcal{A}_q$, the factor $\mu_q^{q-1}(i)$ does not contain a square of letters.*

Proof 4.2. *Assume that $\mu_q^{q-1}(i)$ contains a square of letters. Then there exists $w_1, w_2 \in F(\mathbf{t}_q)$ and $j \in \mathcal{A}_q$ such that $\mu_q^{q-1}(i) = w_1jjw_2$. Therefore, w_1j and jw_2 are respectively images of letters by μ_q^{q-1}. The letter whose the image by μ_q^{q-1} admits j as suffix is $j+q-1$ because the last letter of $\mu_q^{q-1}(j+q-1)$ is $Eq^{(q-1)^2}(j+q-1) = j$ by Proposition 4.1. Then, $w_1j = \mu_q^{q-1}(j+q-1)$ and $jw_2 = \mu_q^{q-1}(j)$. So, $\mu_q^{q-1}(i)$ can be written as:*

$$\begin{aligned} \mu_q^{q-1}(i) &= \mu_q^{q-1}(j+q-1)\mu_q^{q-1}(j) \\ &= \mu_q^{q-1}((j+q-1)j). \end{aligned}$$

The equality $\mu_q^{q-1}(i) = \mu_q^{q-1}((j+q-1)j)$ is impossible because,

$$|\mu_q^{q-1}(i)| \neq |\mu_q^{q-1}((j+q-1)j)|.$$

Consequently, $\mu_q^{q-1}(i)$ does not contain a square of letters.

Lemma 4.2. *For all $(i,j) \in (\mathcal{A}_q \times (\mathcal{A}_q \backslash \{E_q(i)\}))$, the factor $\mu_q^{q-1}(ij)$ does not contain a square of letters.*

Proof 4.3. *Let $(i,j) \in \mathcal{A}_q \times (\mathcal{A}_q \backslash \{E_q(i)\})$. One has:*

$$\begin{aligned} \mu_q^{q-1}(ij) &= \mu_q^{q-1}(i)\mu_q^{q-1}(j) \\ &= (\mu_q^{q-1}(i)(i+1)^{-1})(i+1)j(j^{-1}\mu_q^{q-1}(j)) \end{aligned}$$

because by Proposition 4.1, the last letter of $\mu_q^{q-1}(i)$ is $E_q^{(q-1)^2}(i) = i+1$. Therefore, $\mu_q^{q-1}(ij)$ contains a square of letters if and only if $j = E_q(i)$. That is impossible since $j \neq E_q(i)$ by hypothesis. Consequently, $\mu_q^{q-1}(ij)$ does not contain a square of letters.

Remark 4.3. *The notation i^{-1} in a factor consists to delete the letter i in the factor. For example: $1^{-1}(120201012)2^{-1} = 2020101$.*

Proposition 4.2. *Let w be a separator factor of $\mathbf{t}_q$. Then,*

$$|w| \in \{q^{q-1}-2, 2(q^{q-1}-1)\}.$$

Proof 4.4. *From any square of letter, we construct w using μ_q untill the following square. As $E_q(F_n(t_q)) = F_n(\mathbf{t}_q)$, it suffices us to treat the case of a single square since the others cases can be obtained by successive application of E_q. Consider the square 00.*
Let w_1, w_2 and u be three factors of $\mathbf{t}_q$ such that $u = w_1 00 w_2$. Observe that 00 comes from of $\mu_q(10)$. Then u can be written as $u = w_1'\mu_q(10)w_2'$. As the factor 10 begins with the last letter of the image of 2 and ends by the first letter of the image of 0, then w can be written $w = w_1''\mu_q(\mu_q(2)\mu_q(0))w_2''$. Let us put

$v_1 = w_1''\mu_q(\mu_q(2))$ and $v_2 = \mu_q(\mu_q(0))w_2''$. As 0 is right q-prolongable then $\mu_q^2(0i) \in F(\mathbf{t}_q), \forall i \in \mathcal{A}_q$. We distinguish 3 cases.

Case 1: *$i = 1$. By applying μ_q^{q-3} to $\mu_q^2(01)$ we obtain:*

$$\begin{aligned}\mu_q^{q-3}(\mu_q^2(01)) &= \mu_q^{q-1}(01)\\ &= \mu_q^{q-1}(0)\mu_q^{q-1}(1).\end{aligned}$$

According to Proposition 4.1, the last letter of $\mu_q^{q-1}(i)$ is $Eq^{(q-1)^2}(0) = E_q(0) = 1$. Therefore, we have:

$$\mu_q^{q-1}(01) = (\mu_q^{q-1}(0)1^{-1})11(1^{-1}\mu_q^{q-1}(1)).$$

The factor $\mu_q^{q-1}(0)$ does not contain square of letter by Lemma 4.1. Thus, after the square 00 the following square is 11. As a result, the separator factor is $0^{-1}\mu_q^{q-1}(0)1^{-1}$ and its length is $q^{q-1} - 2$.

Case 2: *For $i = q-1$ we have $\mu_q^2(0(q-1)) \in F(\mathbf{t}_q)$. Observe that the only right extension of $0(q-1)$ is 0. By applying μ_q^{q-3} to $\mu_q^2(0(q-1)0)$ we obtain:*

$$\begin{aligned}\mu_q^{q-3}(\mu_q^2(0(q-1)0)) &= \mu_q^{q-1}(0(q-1)0)\\ &= \mu_q^{q-1}(0(q-1))\mu_q^{q-1}(0).\end{aligned}$$

Proposition 4.1 ensures that the last letter of $\mu_q^{q-1}(q-1)$ is $Eq^{(q-1)^2}(q-1) = 0$. Thus, we have:

$$\mu_q^{q-1}(0(q-1)0) = (\mu_q^{q-1}(0(q-1))0^{-1})00(0^{-1}\mu_q^{q-1}(0)).$$

The factor $\mu_q^{q-1}(0(q-1))$ does not contain a square of letter by Lemma 4.2. Thus, after the square 00 the following is 00, the separator factor is $0^{-1}\mu_q^{q-1}(0(q-1))0^{-1}$ and its length is $2(q^{q-1}-1)$.

Case 3: *For any $i \in \mathcal{A}_q\backslash\{1, q-1\}$, $\mu_q^2(0i) \in F(\mathbf{t}_q)$. As i is the beginning of a image of letter, it extends to the right by j with $j = E_q(i)$. Thus, $\mu_q^2(0ij) \in F(\mathbf{t}_q)$. By applying μ_q^{q-3} to $\mu_q^2(0ij)$ we obtain:*

$$\begin{aligned}\mu_q^{q-3}(\mu_q^2(0ij)) &= \mu_q^{q-1}(0ij)\\ &= \mu_q^{q-1}(0i)\mu_q^{q-1}(j).\end{aligned}$$

According to Proposition 4.1, the last letter of $\mu_q^{q-1}(i)$ is $Eq^{(q-1)^2}(i) = E_q(i) = j$. Thus, we have

$$\mu_q^{q-1}(0ij) = (\mu_q^{q-1}(0i)j^{-1})jj(j^{-1}\mu_q^{q-1}(j)).$$

The factor $\mu_q^{q-1}(0i)$ does not contain a square of letters by Lemma 4.2. Thus, after the square 00 the following square is jj with $j \in [\![1, q-1]\!]\backslash\{2\}$. Consequently, the separator factor is $0^{-1}\mu_q^{q-1}(0i)j^{-1}$ and its length is $2(q^{q-1}-1)$.

We have the following Corollary.

Corollary 4.1. *Let w be a separator factor of $\mathbf{t}_q$. If w separates ii and jj then $j = i$ or $j = E_q^k(i)$, $k \in [\![1, q-1]\!]\backslash\{2\}$.*

(i) If $|w| = q^{q-1} - 2$, then w separates two distinct squares. The separator factors of length $q^{q-1} - 2$ are given by the set:

$$S = \{i^{-1}\mu_q^{q-1}(i)j^{-1} : i, j \in \mathcal{A}_q \text{ and } j = E_q(i)\}.$$

(ii) If $|w| = 2(q^{q-1} - 1)$ then w separates either two identical squares or two distinct squares .

- *The separator factors of length $2(q^{q-1}-1)$ which separate two identical squares are given by the set:*

$$S' = \{i^{-1}\mu_q^{q-1}(i(i+q-1))i^{-1} : i \in \mathcal{A}_q\}.$$

- *The separator factors of length*

$2(q^{q-1}-1)$ *which separate two distinct squares are given by:*

$$S'' = (\bigcup_{k=2}^{q-2}\{i^{-1}\mu_q^{q-1}(im)j^{-1} : i \in \mathcal{A}_q,$$

$$m = E_q^k(i) \text{ and } j = E_q(m)\}) \bigcup \{i^{-1}$$

$$\mu_q^{q-1}(ii)E_q(i)^{-1} : i \in \mathcal{A}_q\}.$$

Proof 4.5. *The proof of this corollary is inspired by that of Proposition* 4.2.

Proposition 4.3. *Let w be a separator factor of length $2(q^{q-1}-1)$. Then w is preceded (resp. followed) by a separator factor of length $q^{q-1}-2$.*

Proof 4.6. *As $E_q(F_n(\mathbf{t}_q)) = F_n(\mathbf{t}_q)$, it suffices to treat the case of a single square. Let w be a separator factor preceded by* 00*. According to Corollary* 4.1*, w takes one of these following forms:*

$$w = 0^{-1}\mu_q^{q-1}(00)1^{-1}$$

or

$$w = 0^{-1}\mu_q^{q-1}(0m)j^{-1}, m \in \mathcal{A}_q\backslash\{1\} \text{and}\, j = E_q(m).$$

Observe that the factor 00 *(resp.* $0m$*) admits a unique left extension and a unique right extension. Therefore,* $(q-1)001$ *and* $(q-1)0mj$ *are factors of* $\mathbf{t}_q$*. By applying* μ_q^{q-1} *to* $(q-1)0mj$ *we obtain:*

$$\mu_q^{q-1}((q-1)0mj) = (\mu_q^{q-1}(q-1)0^{-1})00(0^{-1}\mu_q^{q-1}$$

$$(0m)j^{-1})jj(j^{-1}\mu_q^{q-1}(j)).$$

It follows that $\mu_q^{q-1}(0m)$ is preceded (resp. followed) by:
$(q-1)^{-1}\mu_q^{q-1}(q-1)0^{-1}$ *(resp.* $j^{-1}\mu_q^{q-1}(j)(j+1)^{-1}$*) with $j = E_q(m)$.*
Similarly, we show that $0^{-1}\mu_q^{q-1}(00)1^{-1}$ *is preceded (resp. followed) by* $(q-1)^{-1}\mu_q^{q-1}(q-1)0^{-1}$ *(resp.* $1^{-1}\mu_q^{q-1}(1)2^{-1}$*).*

5 Conclusion

Ultimately, we made a generalized study of the separator and bispecial factors in the generalized Thue-Morse word. Then, we have shown that the only bispecial factor of $\mathbf{t}_q$ are the q-prolongable bispecial and the biprolongable bispecial factors. The bispecial factors present a great interest in the study of the return words of word $\mathbf{t}_q$. As for separator factors they are used in the determination of maximal return time of a word. However, the genelization of some results on the separator factors seen in [15] remains current. Results of this paper will be useful in the study of recurrence function of $\mathbf{t}_q$.

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