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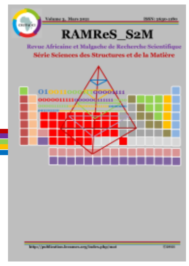

# Worst value and robust value for general optimization problem under uncertainty data

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***Abstract*:**
In this paper we consider general form of a optimization problem in face of data uncertainty. We determine a necessary and sufficient condition ensuring the equality between the worst value and the robust value of this uncertain problem with attainment of the worst value. And we deduce robust strong duality and robust stable strong duality property.

***Keywords*:** Uncertain optimization problem; Robust counterpart; Robust value; Worst value; Robust strong duality.
**Mathematics Subject Classification:** 90C17, 90C31, 90C46

## 1 Introduction

In the real-world optimization problems, we are frequently opposed to the uncertainty of data [1-3]. Some approaches have been developed to resolve these uncertain problems such as robust and stochastic methods [4-10]. Our focus in this paper is about robust optimization. We establish the equality between the value of the robust counterpart and the worst value [11] for a more general form of optimization problem in the face of data uncertainty, with attainment of the worst value.
Consider the following convex optimization problem:

$$(P)\ \inf_x f(x) \text{ s.t. } x \in X$$

where $X$ is a locally convex Hausdorff topological vector space and $f : X \to \mathbb{R} \cup \{+\infty\}$ a proper lower semicontinuous convex function. This problem can be embedded into a family of parameterized problems (see [12]):

$$(P'_y)\ \inf_x \phi(x,y) \text{ s.t. } x \in X,$$

where $y \in Y$; $Y$ is a locally convex Hausdorff topological vector space and, $\phi : X \times Y \to \mathbb{R} \cup \{+\infty\}$ a function satisfying $\phi(x, 0) = f(x)$.
In the face of data uncertainty, $(P'_y)$ can be written:

$$(P_y)\ \inf_x \phi_u(x,y) \text{ s.t. } x \in X$$

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where $\phi_u : X \times Y \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous convex function and $u$ is the uncertain parameter which belongs to the uncertainty set $\mathcal{U}$.
The robust counterpart of the problem $(P_0)$ is the deterministic optimization problem (see [1, 5, 13]).

$$(RP) \quad \inf_x \sup_u \{\phi_u(x,0) : \ u \in \mathcal{U}\} \text{ s.t. } x \in X.$$

The value of the problem $(RP)$ is the robust value of the problem $(P_0)$.
The worst value of the problem $(P_0)$ is the value of the following problem (see [11]).

$$(Q) \quad \sup_u \inf_x \{\phi_u(x,0) : \ x \in X\} \text{ s.t. } u \in \mathcal{U}.$$

The equality between the robust value and the worst value is in someway the well know property "minimax Theorem" [14]. This equality has been proven in [11] in the particular case of uncertain conical convex optimization problem. In this case,

$$\phi_u(x,0) = (f + i_{H_u})(x).$$

where $H_u = \{x \in X \ : \ g_u(x) \in -S \}$, $g_u : X \longrightarrow Y$ is $S$-epi-closed convex and $i_{H_u}$ is the indicator function of $H_u$. So, we generalize the result in [11].
For each fixed $u \in \mathcal{U}$, the conjugate dual problem [15–20] of $(P_0)$ is given by:

$$\sup_{y^*} \{-\phi_u^*(0,y^*)\} \qquad \text{s.t.} \qquad y^* \in Y^*.$$

The optimistic dual of $(P_0)$ is the problem,

$$(DP) \ \sup_u \sup_{y^*} \{-\phi_u^*(0,y^*) : \ y^* \in Y^*\} \quad \text{s.t. } u \in \mathcal{U}.$$

As said above, the focus in this paper is to establish the equality between the value of $(RP)$ and the value of $(Q)$. So, we will be able to deduce robust strong duality and robust stable strong duality property.

The paper is organized as follows. The next section contains some necessary preliminary results of convex analysis that will be used later in the paper. In section 3, we establish the equality between the worst value and the robust value of the uncertain problem $(P_0)$ with attainment of the worst value under necessary and sufficient condition. We establish in the section 4, the robust strong duality and the robust stable strong duality property for $(P_0)$.

# 2 Preliminaries

Let $X$ be a locally convex Hausdorff topological vector space and $f : X \to \mathbb{R} \cup \{+\infty\}$ a function. The dual space of $X$ is denoted by $X^*$. It is known that the space $X^*$ endowed with the weak* topology is a locally convex Hausdorff space. The effective domain and the epigraph of the function $f$ are respectively defined by $\text{dom} f := \{x \in X : f(x) < +\infty\}$ and $\text{epi} f := \{(x,r) \in X \times \mathbb{R} : f(x) \leq r\}$. If $\text{dom} f \neq \emptyset$, we say $f$ is proper. $f$ is a lower semicontinuous function if and only if $\text{epi} f$ is closed. $f$ is say to be convex if and only if $\text{epi} f$ is convex. The Legendre-Fenchel conjugate function of $f$ denoted $f^* : X^* \to \overline{\mathbb{R}}$, is defined by $f^*(x^*) = \sup_{x \in X} \{\langle x^*, x\rangle - f(x)\}$ for all $x^* \in X^*$. It is known that $f^*$ is a proper weak* lower semicontinuous convex function if $f$ is a proper lower semicontinuous convex function.
Given a subset $A \subset X$, we denote by $\text{co}(A)$ the convex hull of $A$, $\overline{A}$ its closure, $\overline{\text{co}}(A)$ its closed convex hull. On the dual space $X^*$ we only consider the week* topology and for any subset $B$ of $X^*$ we simply denote by $\overline{B}$ the week* closure of $B$. Given $A, B$ two subsets of $X$, we say that $A$ is closed regarding $B$ if $\overline{A} \cap B = A \cap B$ ( [15]). $A$ is said to be closed convex regarding $B$ if $\overline{\text{co}}(A) \cap B = A \cap B$ ( [21]).
Given $E \subset \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, we write $\min E$ (respectively $\max E$) instead of $\inf E$ (respectively $\sup E$) when the infimum (respectively supremum) of $E$ is attained. The set of all proper convex lower semi-continuous extended real-valued functions defined on $X$ is denoted by $\Gamma(X)$.
The following Lemma will be useful

later.

**Lemma 1.** *[11, 15] Let $f_i : X \to \mathbb{R} \cup \{+\infty\}$, $i \in I$, be proper lower semicontinuous convex functions on $X$, where $I$ is an arbitrary index set. Suppose that there exists $x_0 \in X$ such that $\sup_{i\in I} f_i(x_0) < +\infty$. Then*

$$\text{epi}\left(\sup_{i\in I} f_i\right)^* = \overline{co}\bigcup \text{epi} f_i^*$$

*where $\sup_{i\in I} f_i : X \to \mathbb{R}\cup\{+\infty\}$ is defined by $(\sup_{i\in I} f_i)(x) = \sup_{i\in I} f_i(x)$ for all $x \in X$.*

# 3 Worst value and robust value

In this section we establish the equality between the worst value and the robust value of the uncertain problem $(P_0)$ with attainment of the worst value,
i.e, we show that :

$$\inf_{x\in X}\sup_{u\in\mathcal{U}} \phi_u(x,0) = \max_{u\in\mathcal{U}}\inf_{x\in X} \phi_u(x,0).$$

Let's remember that the problem $(P_0)$ is defined by,

$$(P_0) \quad \inf \phi_u(x,0) \quad \text{s.t.} \quad x \in X,$$

where $\phi_u : X \times Y \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous convex function and u is an uncertain parameter which belongs to an uncertainty set $\mathcal{U}$.
Let $F = \bigcap_{u\in U} \text{dom}\phi_u(.,0)$ and let's define the function $p : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ by $p = \sup_{u\in\mathcal{U}} \phi_u(.,0)$.

**Remark 1.** *We have* $\text{dom}p = F$ *and* $\inf(RP) = \inf(P)$.

**Proposition 1.** *It holds that*

$$\sup(Q) \leq \inf(RP)$$

*Proof.* We have :

$$\inf_{x\in X} \phi_u(x,0) \leq \phi_u(x,0), \qquad \forall u \in \mathcal{U}$$

$$\sup_{u\in\mathcal{U}}\inf_{x\in X} \phi_u(x,0) \leq \sup_{u\in\mathcal{U}} \phi_u(x,0),$$

$$\sup_{u\in\mathcal{U}}\inf_{x\in X} \phi_u(x,0) \leq \inf_{x\in X}\sup_{u\in\mathcal{U}} \phi_u(x,0),$$

then $\sup(Q) \leq \inf(RP)$.

□

Let us consider the opposite of the problem $(Q)$ namely :

$$(-Q) \quad \inf_u \sup_{x\in X} (-\phi_u(x,0)) \quad s.t.\ u \in \mathcal{U}.$$

The perturbation of the objective function of $(-Q)$ by adding a linear continuous form leads to define the function, $q : X^* \longrightarrow \mathbb{R}$ by :

$$\begin{aligned} q(x^*) &:= \inf_{u\in\mathcal{U}}\sup_{x\in X}\{\langle x, x^*\rangle - \phi_u(x,0)\} \\ &= \inf_{u\in\mathcal{U}} \phi_u^*(x^*,0). \end{aligned}$$

As $\phi_u^{**}(.,0) \leq \phi_u(.,0)$ then,

$$q^* = \sup_{u\in\mathcal{U}} \phi_u^{**}(.,0) \leq \sup_{u\in\mathcal{U}} \phi_u(.,0) = p,$$

therefore $p^* \leq q^{**} \leq q$.

**Lemma 2.** *Assume that $\phi_u(.,0) \in \Gamma(X)$, for all $u \in \mathcal{U}$ and $F \neq \emptyset$. Then*

$$\text{epi}p^* = \overline{co}\left(\bigcup_{u\in\mathcal{U}} \text{epi}\phi_u^*(.,0)\right). \qquad (1)$$

*Proof.* As $\phi_u(.,0) \in \Gamma(X)$, and $F = \bigcap_{u\in U} \text{dom}\phi_u(.,0) \neq \emptyset$, then $\text{dom}p = \text{dom}\left(\sup_{u\in\mathcal{U}} \phi_u(.,0)\right) \neq \emptyset$. By taking Lemma 1 into consideration, it yields

$$\text{epi}p^* = \text{epi}(\sup_{u\in\mathcal{U}} \phi_u(.,0))^* = \overline{\text{co}}\left(\bigcup_{u\in\mathcal{U}} \text{epi}\phi_u^*(.,0)\right).$$

□

**Example 1.** *Let us consider the uncertain conical convex optimization problem:*

$$(\mathcal{P}) \quad \inf_x f(x) \quad s.t.\ g_u(x) \in -S,\ x \in X$$

*where $S \subset Y$ is a nonempty closed convex cone and for each $u \in \mathcal{U}$, $g_u : X \longrightarrow Y$ is $S$-epi-closed convex. Then, appointing $H_u = \{x \in X \ :\ g_u(x) \in -S\}$ and the indicator function of $H_u$, $i_{H_u}$, The problem $(\mathcal{P})$ can be written:*

$$(\mathcal{P}) \qquad \inf_x \phi_u(x,0) \qquad s.t.\ \ x \in X$$

*with,*

$$\phi_u(.,0) : X \longrightarrow \mathbb{R} \cup \{+\infty\}, \quad \phi_u(x,0) = (f + i_{H_u})(x).$$

*So, the problem $(\mathcal{P})$ which has been considered in [11] is a particular case of $(P_0)$. Let us consider the function $p : X \longrightarrow \mathbb{R} \cup \{+\infty\}$, $p(x) = \sup\limits_{u\in\mathcal{U}} \phi_u(x,0) = \sup\limits_{u\in\mathcal{U}} (f + i_{H_u})(x)$.*

*Let us consider the condition:*

$$(C) \qquad \begin{cases} f \in \Gamma(X) \\ (\bigcap_{u\in\mathcal{U}} domg_u) \bigcap domf \neq \emptyset \\ g_u \quad is\ S-level-closed\ convex, \\ \forall u \in \mathcal{U}. \end{cases}$$

*If the condition $(C)$ holds, then $\sup\limits_{u\in\mathcal{U}}(f + i_{H_u}) = \sup\limits_{u\in\mathcal{U}} \phi_u(.,0) \in \Gamma(X)$.*

*Therefore, by using the Lemma 2, one have:*

$$epip^* = \overline{co}\left(\bigcup_{u\in\mathcal{U}} epi\phi_u^*(.,0)\right) = \overline{co}\left(\bigcup_{u\in\mathcal{U}} epi(f + i_{H_u})^*\right).$$

**Theorem 1.** *Assume that $\phi_u(.,0) \in \Gamma(X)$, for all $u \in \mathcal{U}$ and $F \neq \emptyset$. For $x^* \in X^*$ the following statements are equivalent :*

*i)* $p^*(x^*) = \min_{u\in\mathcal{U}} \phi_u^*(x^*,0)$;

*ii)* $\bigcup\limits_{u\in\mathcal{U}} \mathrm{epi}\phi_u^*(.,0)$ *is weak*$^*$*-closed convex regarding* $\{x^*\} \times \mathbb{R}$.

*Proof.* Since dom$p \neq \emptyset$, then $p^*(x^*) \neq -\infty$.
If $p^*(x^*) = +\infty$ then, $i)$ holds, because $p^* \leq q$. And applying Lemma 1 one gets:

$$\overline{\mathrm{co}}(\bigcup_{u\in\mathcal{U}} \mathrm{epi}\phi_u^*(.,0)) \bigcap (\{x^*\} \times \mathbb{R}) = \mathrm{epi}p^* \bigcap (\{x^*\} \times \mathbb{R}) = \emptyset$$

then, $ii)$ holds too.
Assume now that $p^*(x^*) \in \mathbb{R}$.
Let us prove that $ii) \Rightarrow i)$.
By Lemma 1 it holds that

$$(x^*, p^*(x^*)) \in \mathrm{epi}p^* \bigcap (\{x^*\} \times \mathbb{R}) = \overline{\mathrm{co}}(\bigcup_{u\in\mathcal{U}} \mathrm{epi}\phi_u^*(.,0)) \bigcap (\{x^*\} \times \mathbb{R}).$$

As $ii)$ holds, then

$$(x^*, p^*(x^*)) \in (\bigcup_{u\in\mathcal{U}} \mathrm{epi}\phi_u^*(.,0)) \bigcap (\{x^*\}\times\mathbb{R}).$$

So, there exists $\overline{u} \in \mathcal{U}$ such that :

$$\inf_u \phi_u^*(x^*,0) = q(x^*) \leq \phi_{\overline{u}}^*(x^*,0) \leq p^*(x^*).$$

Since $p^*(x^*) \leq q(x^*)$ we get $i)$.
Let us prove now $i) \Rightarrow ii)$.
Let $(x^*, r) \in \overline{\mathrm{co}}(\bigcup\limits_{u\in\mathcal{U}} \mathrm{epi}\phi_u^*(.,0))$.
Applying Lemma 1, we have $p^*(x^*) \leq r$.
By $i)$ there exists $\overline{u} \in \mathcal{U}$ such that : $p^*(x^*) = \phi_{\overline{u}}^*(x^*,0)$ and finally $(x^*, r) \in \mathrm{epi}\phi_{\overline{u}}^*(.,0)$ and we have done. □

**Corollary 1.** *Assume that $\phi_u(.,0) \in \Gamma(X)$, for all $u \in \mathcal{U}$ and $F \neq \emptyset$. Then the following statements are equivalent :*

*i)* $\max_{u\in\mathcal{U}} \inf(P_0) = \inf(RP)$;

*ii) the set* $\bigcup\limits_{u\in\mathcal{U}} \mathrm{epi}\phi_u^*(.,0)$ *is weak*$^*$*-closed convex regarding* $\{0_{X^*}\} \times \mathbb{R}$.

We have,

$$\begin{aligned} p^*(0_{X^*}) &= \sup_{x\in X}\{-\sup_{u\in\mathcal{U}} \phi_u(x,0)\} \\ &= -\inf_{x\in X} \sup_{u\in\mathcal{U}} \phi_u(x,0) \\ &= -\inf(p) = -\inf(RP) \quad (*). \end{aligned}$$

We also have,

$$\begin{aligned} \forall u \in \mathcal{U}, \quad \phi_u^*(0_{X^*},0) &= \sup_{x\in X}\{-\phi_u(x,0)\} \\ &= -\inf_{x\in X}\{\phi_u(x,0)\}, \end{aligned}$$

therefore:

$$\begin{aligned} \min_{u\in\mathcal{U}} \phi_u^*(0_{X^*},0) &= -\max_{u\in\mathcal{U}} \inf_{x\in X}\{\phi_u(x,0)\} \\ &= -\max_{u\in\mathcal{U}} \inf(P_0) \quad (**). \end{aligned}$$

By $(*)$ and $(**)$, one has $\inf(RP) = \max_{u\in\mathcal{U}} \inf(P_0)$ is equivalent to the statement "the set $\bigcup\limits_{u\in\mathcal{U}} \mathrm{epi}\phi_u^*(.,0)$ is weak$^*$-closed convex regarding $\{0_{X^*}\} \times \mathbb{R}$" according to the Theorem 1.

**Corollary 2.** *Assume that $\phi_u(.,0) \in \Gamma(X)$, for all $u \in \mathcal{U}$ and $F \neq \emptyset$. Then the following statements are equivalent :*

*i)* $-\infty < p^*(x^*) = \min_{u\in\mathcal{U}} \phi_u^*(x^*,0) \leq +\infty$,

$$\forall x^* \in X^*;$$

*ii) the set* $\bigcup_{u\in\mathcal{U}} \text{epi}\phi_u^*(.,0)$ *is weak*$^*$*-closed convex.*

*Proof.* The result follows from Theorem 1 because the set $\bigcup_{u\in\mathcal{U}} \text{epi}\phi_u^*(.,0)$ is weak$^*$-closed convex if and only if it is weak$^*$-closed convex regarding $\{x^*\}\times\mathbb{R}$ for all $x^* \in X^*$. □

# 4 Worst value and robust strong duality property

For each fixed $u \in \mathcal{U}$, the conjugate dual problem of $(P_0)$ is given by [22]:

$$(D_u) \qquad \max_{y^*}\{-\phi_u^*(0,y^*)\} \ s.t.\ y^* \in Y^*.$$

The optimistic dual of the uncertain problem $(P_0)$ is given by :

$$(DP) \qquad \sup_u \sup_{y^*}\{-\phi_u^*(0,y^*) : \quad y^* \in Y^*\} \quad \text{s.t.} \quad u \in \mathcal{U}.$$

Robust strong duality property is holds if the values of the robust counterpart and the optimistic dual coincide with dual attainment, i.e.

$$\inf(RP) = \max(DP).$$

**Proposition 2.** *It holds that :*

$$\sup_u \sup_{y^*}\{-\phi_u^*(0,y^*)\} \leq \sup_u \inf_x\{\phi_u(x,0)\}$$

*i.e*

$$\sup(DP) \leq \sup(Q).$$

*Proof.* For each fixed $u \in \mathcal{U}$, the weak duality between $(P_0)$ and $(D_u)$ holds i.e

$$\sup_{y^*\in Y^*}\{-\phi_u^*(0,y^*)\} \leq \inf(P_0).$$

By taking the supremum of the terms in both sides of it, the desired conclusion follows. □

We give robust version of the weak duality.

**Lemma 3** (Robust Weak Duality)**.** *If* $\phi_u : X \times Y \longrightarrow \mathbb{R}\cup\{+\infty\}$ *is a proper lower semicontinuous and convex function for any* $u \in \mathcal{U}$*. Then,*

$$\inf_{x\in X} \sup_{u\in\mathcal{U}} \phi_u(x,0) \geq \sup_{u\in\mathcal{U}} \sup_{y^*\in\mathcal{Y}^*} \{-\phi_u^*(0,y^*)\}.$$

*Proof.* The juxtaposition of Proposition 1 and 2 give us the result. □

One says that robust strong duality holds for the problem $(P_0)$, whenever the values of the robust counterpart and the optimistic dual coincide with dual attainment [11, 22], i.e :

$$\inf(RP) = \max(DP).$$

**Proposition 3.** *If robust strong duality for* $(P_0)$ *holds then,*

$$\inf(RP) = \max(Q).$$

*Proof.* With the Proposition 1 and Proposition 2 we have:

$$\max(DP) = \sup(Q) = \inf(RP).$$

Consequently, there exists $(\overline{u},\overline{y}) \in \mathcal{U}\times Y^*$ such that :

$$\begin{aligned} -\phi_{\overline{u}}^*(0,\overline{y}^*) &= \inf(RP) \\ &= \sup(Q) \\ &\geq \inf(P_{0,\overline{u}}) \\ &\geq -\phi_{\overline{u}}^*(0,\overline{y}^*), \end{aligned}$$

where the last inequality follows from the weak duality between $(P_{0,\overline{u}})$ and $(D_{\overline{u}})$. Thus,

$$\inf(RP) = \sup(Q) = \inf(P_{0,\overline{u}}),$$

so, $\sup(Q)$ is attain. □

We need following statement to establish robust strong duality property for $(P_0)$.

**Lemma 4** ( [15])**.** *Let* $\phi : X \times Y \longrightarrow \overline{\mathbb{R}}$ *be a proper, convex and lower semicontinuous function such that* $0 \in \text{Pr}_Y(\text{dom}\phi)$ *and* $V$ *a nonempty subset of* $X^*$*. Then the following statements are equivalent :*

*(i)* $(\phi(.,0))^*(x^*) := \sup_{x\in X}\{\langle x^*,x\rangle - \phi(x,0)\} = \min_{y^*\in Y^*} \phi^*(x^*,y^*), \quad \forall x^* \in V$;

*(ii)* $\mathrm{Pr}_{X^*\times\mathbb{R}}(\mathrm{epi}\phi^*)$ *is weak*-closed regarding the set* $V\times\mathbb{R}$.

**Remark 2.** *We see that* $F\neq\emptyset \Longleftrightarrow 0\in Pr_Y(dom\phi_u)$.

**Proposition 4.** *If* $F\neq\emptyset$ *and for all* $u\in\mathcal{U}$, $Pr_{X^*\times\mathbb{R}}(\mathrm{epi}\phi_u^*)$ *is closed regarding the set* $\{0_{X^*}\}\times\mathbb{R}$ *then :*

$$\sup(DP)=\sup(Q).$$

*Proof.* With the Lemma 4, for all $u\in\mathcal{U}$,

$$\sup_{x\in X}\{-\phi_u(x,0)\}=\min_{y^*\in Y^*}\phi_u^*(0,y^*),$$

then,

$$\inf_{x\in X}\{\phi_u(x,0)\}=\max_{y^*\in Y^*}\{-\phi_u^*(0,y^*)\},$$

so,

$$\sup(DP)=\sup(Q).$$

□

**Theorem 2.** *Assume* $\phi_u(.,0)\in\Gamma(X)$, $\forall u\in\mathcal{U}$ *and* $F\neq\emptyset$.
*If* $\bigcup_{u\in\mathcal{U}}\mathrm{epi}\phi_u^*(.,0)$ *is weak*-closed convex regarding* $\{0_{X^*}\}\times\mathbb{R}$ *and if it exists* $\overline{u}\in\mathcal{U}$ *such that* $\mathrm{Pr}_{X^*\times\mathbb{R}}(\mathrm{epi}\phi_{\overline{u}}^*)$ *is weak*-closed convex regarding* $\{0_{X^*}\}\times\mathbb{R}$ *then robust strong duality property holds.*

*Proof.* As $\bigcup_{u\in\mathcal{U}}\mathrm{epi}\phi_u^*(.,0)$ is weak*-closed convex regarding $\{0_{X^*}\}\times\mathbb{R}$ then we have from Corollary 1, $\inf(RP)=\max(Q)$. So, there exists $\overline{u}\in\mathcal{U}$ such that $\inf(RP)=\inf(P_{0,\overline{u}})$. As $\mathrm{Pr}_{X^*\times\mathbb{R}}(\mathrm{epi}\phi_{\overline{u}}^*)$ is weak*-closed convex regarding $\{0_{X^*}\}\times\mathbb{R}$ it follows from the Lemma 4 that $\inf(RP)=\max(D_{\overline{u}})\leq\sup(DP)$. From proposition 2, we have $\sup(DP)\leq\max(Q)=\inf(RP)=\max(D_{\overline{u}})\leq\sup(DP)$. Consequently, $\inf(RP)=\max(DP)$. □

One says that robust stable strong duality holds for $(P_0)$, if for each $x^*\in X^*$, one has:

$$\inf_{x\in X}\sup_{u\in\mathcal{U}}\{\phi_u(x,0)-\langle x^*,x\rangle\}=$$

$$\max_{u\in\mathcal{U}}\max_{y^*\in Y^*}\{-\phi^*(x^*,y^*)\}.$$

Now, let us establish robust stable strong duality for $(P_0)$.

**Corollary 3.** *If* $\phi_u(.,0)\in\Gamma(X)$, *for all* $u\in\mathcal{U}$, $F\neq\emptyset$, *the set* $\bigcup_{u\in\mathcal{U}}\mathrm{epi}\phi_u^*(.,0)$ *is weak*-closed convex and* $\mathrm{Pr}_{X^*\times\mathbb{R}}(\mathrm{epi}\phi^*)$ *is weak*-closed. Then, robust stable strong duality holds for* $(P_0)$.

*Proof.* With corollary 2, we have

$$\inf_{x\in X}\sup_{u\in\mathcal{U}}\{\phi_u(x,0)-\langle x^*,x\rangle\}=$$

$$\max_{u\in\mathcal{U}}\inf_{x\in X}\{\phi_u(x,0)-\langle x^*,x\rangle\},$$

then there exists $\overline{u}\in\mathcal{U}$ such that

$$\inf_{x\in X}\sup_{u\in\mathcal{U}}\{\phi_u(x,0)-\langle x^*,x\rangle\}=$$

$$\inf_{x\in X}\{\phi_{\overline{u}}(x,0)-\langle x^*,x\rangle\}.$$

Lemma 4 gives us

$$\inf_{x\in X}\sup_{u\in\mathcal{U}}\{\phi_u(x,0)-\langle x^*,x\rangle\}=$$
$$\max_{y^*\in Y^*}\{-\phi_{\overline{u}}^*(x^*,y^*)\}\leq$$
$$\sup_{u\in\mathcal{U}}\max_{y^*\in Y^*}\{-\phi_u^*(x^*,y^*)\}.$$

We conclude by using the robust weak duality. □

# 5 Conclusion

We have considered the general form of convex optimization problem under uncertainty :

$$(P_0)\quad \inf_x \phi_u(x,0)\ \text{ s.t. }\ x\in X.$$

We have established equality between the worst value and the robust value of $(P_0)$. Then, we have deduced the robust strong duality property. So, we generalize the results in [11], where the problem considered is in the form:

$$(\mathcal{P})\qquad \inf_x f(x)\quad s.t.\ g_u(x)\in -S,\ x\in X.$$

In particular, the condition

$$\begin{cases}\phi_u(.,0)\in\Gamma(X),\forall u\in\mathcal{U}\text{ and }F\neq\emptyset\\ \bigcup_{u\in\mathcal{U}}\mathrm{epi}\phi_u^*(.,0)\ \text{ is weak*-closed convex}\\ \text{regarding }\ \{0_{X^*}\}\times\mathbb{R}\end{cases}$$

ensuring equality between the worst value and the robust value in this paper is weak than the condition:

$$(\mathcal{H}) \qquad \begin{cases} f \in \Gamma(X) \\ F \cap \operatorname{dom} f \neq \emptyset \\ g_u \ \text{ is C-level-closed,} \forall u \in U \\ \bigcup_{u\in U} \operatorname{epi}(f + i_{F_u})^* \text{ is weak-closed} \\ \ \text{convex regarding } \{0_{X^*}\} \times \mathbb{R}. \end{cases}$$

which has been considered in [11].

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