RAMReS Sciences des Structures et de la Matière Vol. 7, N°2 (2023) 132 - 143
http://publication.lecames.org/index.php/mat ISSN: 2630-1180

Algebra of differential operators on Laurent polynomials rings

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Received: 28 September 2023 / Received in revised form: 12 December 2023 / Accepted: 27 December 2023

***Abstract*:** In this paper, we determine exactly the algebra of differential operators on Laurent polynomials rings over an arbitrary characteristic field k.

***Keywords*:** Differential operators; Weyl algebra; Laurent polynomials rings.
MSC 2020: Primary 13N10; Secondary 16S32.

1 Introduction

In the sequel, $L = k\,[x_1^{\pm 1}, \dots\dots, x_n^{\pm 1}]$ denotes the ring of Laurent polynomials with n variables over a field k and I an ideal of L. Differential operators on a given A-module M, were introduced in 1967 by Grothendieck [1]. Some authors have studied the algebra of differential operators on a given algebra. This is the case of Uma Lyer [2] and William Nathaniel [3] who determined exactly the algebra of differential operators on Hoph algebras and on Stanley-Reisner rings, respectively. As far as we are concerned, we determine exactly the algebra of differential opera tors on Laurent polynomials rings, introduced in 1943 by the mathematician Pierre Alphonse Laurent. In doing so, our work is divided into two parts. We devote the first section to the definitions, the remarkable properties of differential operators and the ring of Laurent polynomials. In the second section, we determine first a sufficient condition so that two differential operators on a Laurent polynomials ring are equal (*Corollary* 3.3). And then, we determine exactly, thanks to this sufficient condition, the algebra of differential operators on L when the characteristic of the field k is zero (*Theorem* 3.4).

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Finally we define in nonzero characteristic, an infinite family of differential operators on L which will generate the algebra of differential operators on L (*Theorem* 3.5).

For the notations used in this paper see Appendix.

2 Ring of Laurent polynomials: Algebra of differential operators

2.1 Laurent polynomials ring

Definition 2.1. *Let R be a commutative ring.*
A Laurent polynomial with coefficients in R is an expression of the form :

$$p(x) = \sum_{k\in\mathbb{Z}} a_k x^k, \quad a_k \in R$$

where only a finite number of the coefficients a_k are different from 0.

The set of Laurent polynomials with coefficients in a commutative ring R is denoted $R\left[x, x^{-1}\right]$ or $R\left[x^{\pm 1}\right]$. This set is provided with a ring structure with the same operations as the ring of polynomials over R, where summation index could take negative values.
In particular, the Laurent polynomials ring is obtained by localization of the ring of polynomials. The multiplicative part of this localization is $S = \{x^n, n \in \mathbb{N}\}$. Therefore, we have the following operations:

$$i) \left(\sum_i a_i x^i\right) + \left(\sum_i b_i x^i\right) = \sum_i (a_i + b_i)\, x^i;$$

$$ii) \left(\sum_i a_i x^i\right) . \left(\sum_j b_j x^j\right) = \sum_k \left(\sum_{i,j:i+j=k} a_i b_j\right) x^k;$$

and the natural structure of R-module allows to define the multiplication by a scalar

$$a \sum_{k\in\mathbb{Z}} a_k x^k = \sum_{k\in\mathbb{Z}} a a_k x^k.$$

Laurent's polynomials can be generalized easily to several indeterminates, the correspon-
ding ring being denoted $R\left[x_i^{\pm 1}\right]$.

Properties 1.

1) *$R[x]$ is a subring of $R\left[x^{\pm 1}\right]$ wich is a subring of the ring of rational fractions $R(x)$.*
2) *The ring $R\left[x^{\pm 1}\right]$ is a Noetherian ring but not an Artinian one.*
3) *If k is a field, then $k\left[x^{\pm 1}\right]$ is an Euclidean ring (as localized of $k(x)$).*

Definition 2.2. (Derivations on Laurent polynomials ring).
Let R be a zero characteristic field.

1) *A derivation on $R\left[x_i^{\pm 1}\right]$ is:*

$$\partial_i : \sum_{\alpha\in\mathbb{Z}^n} \ell_\alpha x^\alpha \longmapsto \sum_i \alpha_i \ell_\alpha x_i^{\alpha_i - 1}.$$

2) *Any derivation on $R\left[x_i^{\pm 1}\right]$ is in the form $\sum\limits_i \ell_i \partial_i$, for all $\ell_i \in R\left[x_i^{\pm 1}\right]$.*

2.2 Algebra of differential operators on a commutative algebra

Definition 2.3. (Differential operators).
Let A be a commutative algebra over a field k, and let M and N be left A-modules. The set of differential operators from M to N is defined inductively by :

$$\mathcal{D}_A(M,N) = \bigcup_{n\in\mathbb{N}} \mathcal{D}_A^n(M,N),$$

where

$$\mathcal{D}_A^0(M,N) = Hom_A(M,N)$$

and for $n \in \mathbb{N}^*$,

$$\mathcal{D}_A^n(M,N) = \{u \in Hom_k(M,N) : [u,a] = ua - au \in \mathcal{D}_A^{n-1}(M,N), \forall a \in A\}.$$

ua and au are elements of $Hom_k(M, N)$ defined by:

$$\forall m \in M,\ ua(m) = u(am) \text{ and } au(m) = a\,(u(m))\,.$$

Any element $u \in \mathcal{D}_A^n(M, N)$ is called differential operator of order n from M to N.

Definition 2.4. (Algebra of differential operators).
Let A be a k-algebra and let M be an A-module. When $M = N = A$, the algebra $\mathcal{D}_A(A)$ that we will denote $\mathcal{D}(A)$, is called the algebra of differential operators on A.

Properties 2.

1) *$End_k(A)$ is a (A, A)-bimodule.*
2) *$End_k(A)$ is a $(\mathcal{D}(A), \mathcal{D}(A))$-bimodule.*
3) *Let $n \geq 0$ be an integer, $\mathcal{D}^n(A)$ is a (A, A)-subbimodule of $End_k(A)$.*
4) *$\mathcal{D}(A)$ is a (A, A)-subbimodule of $End_k(A)$.*

Proof.

1) We know that $End_k(A)$ is an additive group.
   a) Let $\varphi_1,\ \varphi_2$ be the following applications:
   $$\varphi_1 : \begin{array}{c} A \times End_k(A) \longrightarrow End_k(A) \\ (a, u) \longmapsto au \end{array}$$
   and
   $$\varphi_2 : \begin{array}{c} End_k(A) \times A \longrightarrow End_k(A) \\ (u, a) \longmapsto ua \end{array}.$$
   We deduce from φ_1 (respectively from φ_2), that $End_k(A)$ is a left A-module (respectively a right A-module).
   b) In addition, for all $a \in A$ and all $u, v \in End_k(A)$, we have:
   $$(ua) \circ v = u \circ (av)\,.$$
   As a result, according to *a*) and *b*), $End_k(A)$ is a (A, A)-bimodule.
2) Let us show that $End_k(A)$ is a $(\mathcal{D}(A), \mathcal{D}(A))$-bimodule.
   a) Let φ_3 , φ_4 be the following applications:
   $$\varphi_3 : \begin{array}{c} \mathcal{D}(A) \times End_k(A) \longrightarrow End_k(A) \\ (u, v) \longmapsto u \circ v \end{array}$$
   and
   $$\varphi_4 : \begin{array}{c} End_k(A) \times \mathcal{D}(A) \longrightarrow End_k(A) \\ (v, u) \longmapsto v \circ u \end{array}.$$
   We deduce from φ_3 (respectively from φ_4), that $End_k(A)$ is a left $\mathcal{D}(A)$-module (respectively a right $\mathcal{D}(A)$-module).
   b) For all $v_1, v_2 \in End_k(A)$ and all $u \in \mathcal{D}(A)$, we have:
   $$(u \circ v_1) \circ v_2 = u \circ (v_1 \circ v_2)\,.$$
   According to these two points, $End_k(A)$ is a $(\mathcal{D}(A), \mathcal{D}(A))$-bimodule.
3) Let $n \geq 0$ be an integer and show that $\mathcal{D}^n(A)$ is a (A, A)-subbimodule of $End_k(A)$.
   a) We show by induction on the order n, that $\mathcal{D}^n(A)$ is a subgroup of $End_k(A)$. We have:
      i) $0_{End_k(A)} \in \mathcal{D}^n(A)$ and $\mathcal{D}^n(A) \subseteq End_k(A)$.
      ii) Let $u, v \in \mathcal{D}^n(A)$.
      - For $n = 0$, we have $u - v \in \mathcal{D}^0(A)$ because $\mathcal{D}^0(A) = End_A\,(A)\,.$
      - Induction hypothesis. Let $n \in \mathbb{N}^*$ and $q \in \mathbb{N}$ such that $q < n$. If $u, v \in \mathcal{D}^q(A)$, then $u - v \in \mathcal{D}^q(A)$.
      - Suppose $u, v \in \mathcal{D}^n(A)$ and let show that
      $$u - v \in \mathcal{D}^n(A).$$

We have

$$u, v \in \mathcal{D}^n(A) \iff \begin{cases} [u,a] \in \mathcal{D}^{n-1}(A), \forall a \in A \\ [v,a] \in \mathcal{D}^{n-1}(A), \forall a \in A \end{cases}.$$

We deduce by induction hypothesis that:

$$[u-v,a] = [u,a] - [v,a] \in \mathcal{D}^{n-1}(A), \forall a \in A.$$

Yet

$$\mathcal{D}^n(A) = \{w \in End_k(A), [w,a] \in \mathcal{D}^{n-1}(A)\},$$

thus $u - v \in \mathcal{D}^n(A)$.

We conclude that for all $n \in \mathbb{N}$ and all $u, v \in \mathcal{D}^n(A)$,

$$u - v \in \mathcal{D}^n(A).$$

From $i)$ and $ii)$, we deduce that for all $n \in \mathbb{N}$, $\mathcal{D}^n(A)$ is a subgroup of $End_k(A)$.

b) Let $a, b \in A$ and $u \in \mathcal{D}^n(A)$. We have

$$[au, b] = a[u,b].$$

Yet $u \in \mathcal{D}^n(A)$, thus

$$[au, b] = a[u,b] \in \mathcal{D}^{n-1}(A),$$

wich means that $au \in \mathcal{D}^n(A)$. So,

$$\forall (a,n) \in A \times \mathbb{N}, \quad a\mathcal{D}^n(A) \subseteq \mathcal{D}^n(A).$$

c) Likewise we show that

$$\forall (a,n) \in A \times \mathbb{N}, \quad \mathcal{D}^n(A)a \subseteq \mathcal{D}^n(A).$$

Therefore, from 3.a), 3.b) and 3.c), we deduce that for all $n \in \mathbb{N}$, $\mathcal{D}^n(A)$ is a (A,A)-subbimodule of $End_k(A)$.

4) Let us show that $\mathcal{D}(A)$ is a (A,A)-subbimodule of $End_k(A)$.
Let $a \in A$, $u \in \mathcal{D}(A)$.

a) $\mathcal{D}(A)$ is a subgroup of $End_k(A)$ because for all $n \in \mathbb{N}$, $\mathcal{D}^n(A)$ is a subgroup of $End_k(A)$.

b) $u \in \mathcal{D}(A)$ is equivalent to there exists $n \in \mathbb{N}$ such that $u \in \mathcal{D}^n(A)$. So, by 3.b) and 3.c), we have

$$\forall n \in \mathbb{N}, \; au \in \mathcal{D}(A) \text{ and } ua \in \mathcal{D}(A).$$

From 4.a) and 4.b), $\mathcal{D}(A)$ is a (A,A)-subbimodule of $End_k(A)$.

Proposition 2.1.

1) *$id_A \in \mathcal{D}(A)$ and $\mathcal{D}^0(A)$ is identified to A by*

$$\varphi : A \longrightarrow \mathcal{D}^0(A)$$
$$a \longmapsto \varphi_a$$

such that

$$\forall x \in A, \quad \varphi_a(x) = ax.$$

2) $\mathcal{D}^n(A) \subseteq \mathcal{D}^{n+1}(A)$, *for all* $n \in \mathbb{N}$.

3) $\mathcal{D}^m(A).\mathcal{D}^n(A) \subseteq \mathcal{D}^{m+n}(A)$, *for all* $m, n \in \mathbb{N}$.

4) $[\mathcal{D}^m(A), \mathcal{D}^n(A)] \subseteq \mathcal{D}^{m+n-1}(A)$, *for all* $m, n \in \mathbb{N}$.

5) $\mathcal{D}^m(A) + \mathcal{D}^n(A) \subseteq \mathcal{D}^{Max\{m,n\}}(A)$, *for all* $m, n \in \mathbb{N}$.

6) $\mathcal{D}(A)$ *is a subalgebra of* $End_k(A)$.

Proof.

1) It is known that $id_A \in End_A(A)$. Yet $\mathcal{D}^0(A) = End_A(A)$ and $\mathcal{D}^0(A) \subset \mathcal{D}(A)$, thus $id_A \in \mathcal{D}(A)$. Let us show that $\mathcal{D}^0(A)$ is identified to A.
Consider $u \in \mathcal{D}^0(A)$ and $a \in A$. Then $ua, au \in \mathcal{D}^0(A)$ because $\mathcal{D}^0(A)$ is a (A,A)-bimodule. Let

$$\varphi : A \longrightarrow \mathcal{D}^0(A)$$
$$x \longmapsto \varphi_x : A \longrightarrow A$$
$$a \longmapsto xa$$

be a morphism. We have:

$$u \in \mathcal{D}^0(A) \iff [u,x] = ux - xu = 0_{End_k(A)}, \forall x \in A.$$

So,

$$\forall a \in A,\ u(xa) = xu(a) = xau(1_A), \quad (*)$$

wich means that

$$\forall a \in A, \quad u \circ \varphi_x(a) = \varphi_x \circ u(a).$$

In particular, for $a = 1$, $(*)$ becomes

$$\forall x \in A,\ u(x) = xu(1_A) = \varphi_{u(1_A)}(x).$$

So, $u = \varphi_{u(1_A)}$, that is for any element $u \in \mathcal{D}^0(A)$, identifies with a multiplicative morphism φ_b, where $b = u(1_A) \in A$.

2) Let us show by induction that:

$$\mathcal{D}^n(A) \subseteq \mathcal{D}^{n+1}(A), \quad \forall n \in \mathbb{N}.$$

i) For $n = 0$, we have

$$\mathcal{D}^0(A) = End_A(A) \subseteq End_k(A).$$

Let $u \in \mathcal{D}^0(A)$ and $(a,x) \in A^2$. We have:

$$[u,a](x) = u(ax) - au(x) = au(x) - au(x) = 0$$

$(u(ax) = au(x)$ because $u \in End_A(A))$. So,

$$\forall a \in A, [u,a] \in \mathcal{D}^0(A).$$

Yet

$$\mathcal{D}^1(A) = \{u \in End_k(A) : [u,a] \in \mathcal{D}^0(A), \forall a \in A\},$$

thus $\mathcal{D}^0(A) \subseteq \mathcal{D}^1(A)$.

ii) Induction hypothesis.
Let $n \in \mathbb{N}^*$ and $q \in \mathbb{N}$. If $0 \leq q < n$, then $\mathcal{D}^q(A) \subseteq \mathcal{D}^{q+1}(A)$.

iii) Let show that $\mathcal{D}^n(A) \subseteq \mathcal{D}^{n+1}(A)$.
Consider $u \in \mathcal{D}^n(A)$. That is equivalent to

$$u \in End_k(A) \text{ and } [u,a] \in \mathcal{D}^{n-1}(A), \forall a \in A.$$

By Induction hypothesis we have $\mathcal{D}^{n-1}(A) \subseteq \mathcal{D}^n(A)$. So,

$$[u,a] \in \mathcal{D}^n(A), \quad \forall a \in A.$$

Yet

$$\mathcal{D}^{n+1}(A) = \{u \in End_k(A) : [u,a] \in \mathcal{D}^n(A), \forall a \in A\},$$

thus $u \in \mathcal{D}^{n+1}(A)$ and we obtain $\mathcal{D}^n(A) \subseteq \mathcal{D}^{n+1}(A)$.

Conclusion : $\mathcal{D}^n(A) \subseteq \mathcal{D}^{n+1}(A)$, $\forall n \in \mathbb{N}$.

3) Let us show by induction on $r = m+n$ that:

$$\mathcal{D}^m(A).\mathcal{D}^n(A) \subseteq \mathcal{D}^{m+n}(A),\ \forall m,n \in \mathbb{N}.$$

i) For $r = m = n = 0$, we have $\mathcal{D}^0(A).\mathcal{D}^0(A) \subseteq \mathcal{D}^0(A)$ because $\mathcal{D}^0(A) = End_A(A)$.

ii) Induction hypothesis.
Let $n \in \mathbb{N}^*$ and $r, s, t \in \mathbb{N}$. If $0 \leq r < n$ where $r = s + t$, then

$$\mathcal{D}^s(A).\mathcal{D}^t(A) \subseteq \mathcal{D}^r(A).$$

iii) Let us show that for $r = n = s + t$, $\mathcal{D}^s(A).\mathcal{D}^t(A) \subseteq \mathcal{D}^n(A)$.
Let $u \in \mathcal{D}^s(A)$, $v \in \mathcal{D}^t(A)$ and $a \in A$. Obviously, $u \circ v \in End_k(A)$ and we have

$$[u \circ v, a] = u \circ [v,a] + [u,a] \circ v,$$

yet $[u,a] \in \mathcal{D}^{s-1}(A)$ and $[v,a] \in \mathcal{D}^{t-1}(A)$. Thus, by Induction hypothesis we have :

$$u \circ [v,a] \in \mathcal{D}^{s+t-1}(A)$$

and

$$[u,a]\circ v \in \mathcal{D}^{s+t-1}(A).$$

Therefore,

$$\forall a \in A, [u\circ v, a] \in \mathcal{D}^{s+t-1}(A) = \mathcal{D}^{n-1}(A).$$

It follows that $u \circ v \in \mathcal{D}^n(A)$, so $\mathcal{D}^s(A).\mathcal{D}^t(A) \subseteq \mathcal{D}^n(A)$.

We conclude that:

$$\mathcal{D}^m(A).\mathcal{D}^n(A) \subseteq \mathcal{D}^{m+n}(A), \forall m,n \in \mathbb{N}.$$

4) Let us show by induction that

$$[\mathcal{D}^m(A), \mathcal{D}^n(A)] \subseteq \mathcal{D}^{m+n-1}(A),\ \ \forall m,n \in \mathbb{N},$$

with $\mathcal{D}^{-1}(A) = \{0_{End_k(A)}\}$.

i) Let $m = n = 0$ and $u, v \in \mathcal{D}^0(A)$. We have $[u,v] = u\circ v - v\circ u \in \mathcal{D}^0(A)$ because $\mathcal{D}^0(A) = End_A(A)$. So,

$$\forall a \in A, [[u,v],a] = 0_{End_k(A)} \in \mathcal{D}^{-1}(A).$$

We obtain then

$$[\mathcal{D}^0(A), \mathcal{D}^0(A)] \subseteq \mathcal{D}^{-1}(A).$$

ii) Induction hypothesis.
Consider $s \in \mathbb{N}^*$ and $m,n,r \in \mathbb{N}$ such that $r = m+n$. If $0 \le r < s$, then $[\mathcal{D}^m(A), \mathcal{D}^n(A)] \subseteq \mathcal{D}^{r-1}(A)$.

iii) Let us show that for $r = s$, we have $[\mathcal{D}^m(A), \mathcal{D}^n(A)] \subseteq \mathcal{D}^{s-1}(A)$.
Let $(u,v) \in \mathcal{D}^m(A)\times\mathcal{D}^n(A)$ and $a \in A$. We have

$$[a,[u,v]] + [v,[a,u]] + [u,[v,a]] = 0 \text{ (Jacobi relation).}$$

So,

$$\begin{aligned}[[u,v],a] &= [v,[a,u]] + [u,[v,a]] \\ &= -[v,[u,a]] + [u,[v,a]].\end{aligned}$$

Yet

$$[u,a] \in \mathcal{D}^{m-1}(A) \text{ and } [v,a] \in \mathcal{D}^{n-1}(A),\ \ \forall a \in A,$$

thus, by induction hypothesis, we have:

$$\begin{cases} [v,[u,a]] \in \mathcal{D}^{m+n-2}(A), \forall a \in A \\ [u,[v,a]] \in \mathcal{D}^{m+n-2}(A), \forall a \in A \end{cases},$$

because $m+n-1 < s$. It follows that

$$[[u,v],a] \in \mathcal{D}^{m+n-2}(A),\ \ \forall a \in A,$$

which means that

$$[u,v] \in \mathcal{D}^{m+n-1}(A) = \mathcal{D}^{s-1}(A).$$

Consequently, for all $m,n \in \mathbb{N}$,

$$[\mathcal{D}^m(A), \mathcal{D}^n(A)] \subseteq \mathcal{D}^{m+n-1}(A).$$

5) Let us show by induction that:

$$\mathcal{D}^m(A)+\mathcal{D}^n(A) \subseteq \mathcal{D}^{Max\{m,n\}}(A), \forall m,n \in \mathbb{N}.$$

i) For $m = n = 0$, we have:

$$\mathcal{D}^0(A) + \mathcal{D}^0(A) \subseteq \mathcal{D}^0(A)$$

due to the fact that

$$\mathcal{D}^0(A) = End_A(A).$$

ii) Induction hypothesis.
For all $s \in \mathbb{N}^*$ and all $(m,n) \in \mathbb{N}^2$, if $0 \le m,n < s$, then

$$\mathcal{D}^m(A) + \mathcal{D}^n(A) \subseteq \mathcal{D}^{\max\{m,n\}}.$$

iii) Let us show that

$$\mathcal{D}^m(A) + \mathcal{D}^n(A) \subseteq \mathcal{D}^s(A)$$

with $s = \max\{m,n\}$. Consider $a \in A$ and $(u,v) \in \mathcal{D}^m(A) \times \mathcal{D}^n(A)$. We have

$$[u+v,a] = [u,a] + [v,a].$$

Yet

$$[u,a] \in \mathcal{D}^{m-1}(A) \text{ and } [v,a] \in \mathcal{D}^{n-1}(A),$$

thus, by induction hypothesis we have :

$$[u+v,a] \in \mathcal{D}^{Max\{m-1,n-1\}}(A).$$

Yet $\max\{m-1,n-1\} = s-1$, so $u+v \in \mathcal{D}^s(A)$.

It follows that for all $m, n \in \mathbb{N}$,

$$\mathcal{D}^m(A) + \mathcal{D}^n(a) \subseteq \mathcal{D}^{Max\{m,n\}}(A).$$

6) According to 1), 2) and 3), $\mathcal{D}(A)$ is a subring of $End_k(A)$. In addition, for all $\alpha \in k$, $\alpha\mathcal{D}(A) \subseteq \mathcal{D}(A)$. So, the subring $\mathcal{D}(A)$ is a subalgebra of $End_k(A)$.

Proposition 2.2. (cf. [4])
For all k-algebra A, $\mathcal{D}^1(A) = A \oplus Der_k(A)$.

2.3 Weyl algebra

Definition 2.5. *The n-th weyl algebra is an associative, unitary and non-commutative algebra, generated by $2n$ elements $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$, that satisfy the following defining relations:*

1) $[x_i, x_j] = [y_i, y_j] = 0$;

2) $[y_i, x_j] = \delta_{ij}$, *where δ_{ij} is the Kronecker delta for all $i, j \in \{1, \ldots, n\}$.*

The n-th weyl algebra is denoted $A_n(k)$.

The following proposition shows that in characteristic zero, the algebra of differential operators on a polynomial algebra is a Weyl algebra.

Proposition 2.3. (cf. [5])
Let k be a field of characteristic zero. The algebra of differential operators on the polynomial k-algebra $k[x_1, \ldots, x_n]$, is the n-th Weyl algebra:

$$\mathcal{D}(k\left[x_1, .., x_n\right]) \simeq A_n\left(k\right).$$

3 Algebra of differential operators on Laurent polynomials ring

Let $L = k\left[x_1^{\pm 1}, .., x_n^{\pm 1}\right]$ be a Laurent polynomials ring with n variables over a field k and I an ideal of L.

3.1 Equality of two differential operators on a Laurent polynomials ring

In this paragraph, we determine a sufficient condition so that two differential operators of the same order on the ring of Laurent polynomials are equal.

Lemma 1. *Let $m \in \mathbb{N}$.*
If $u \in \mathcal{D}^m(L)$ such that, for all $a \in V^m$, $u\left(a\right) \in I$, then

$$\forall b \in V^{m-1},\ \forall i \in \overline{1,n},\quad \left[u, x_i\right]\left(b\right) \in I.$$

Proof. Indeed for $u \in \mathcal{D}^m(R)$ such, that for all $a \in V^m$, $u\left(a\right) \in I$ and all $b \in V^{m-1}$, we have

$$u\left(x_i b\right) \in I,\ \forall i \in \overline{1,n}$$

because $x_i b \in V^m$ and I is an ideal of R. So,

$$\forall i \in \overline{1,n},\ \left[u, x_i\right]\left(b\right) = u\left(x_i b\right) - x_i u\left(b\right) \in I.$$

Lemma 2. *Let $u, v \in \mathcal{D}(L)$. If $u_+ = v_+$, then for all $\ell \in L_+, \left[u, \ell\right]_+ = \left[v, \ell\right]_+$.*

Proof. Let $u, v \in \mathcal{D}(L)$ such that $u_+ = v_+$, and $\ell_1, \ell_2 \in L_+$. We have

$$\left(\left[u, \ell_1\right]\right)\left(\ell_2\right) = u\left(\ell_1 \ell_2\right) - \ell_1 u\left(\ell_2\right).$$

Yet $u_+ = v_+$ and $\ell_1 \ell_2 \in L_+$. So,

$$\left(\left[u, \ell_1\right]_+\right)\left(\ell_2\right) = v\left(\ell_1 \ell_2\right) - \ell_1 v\left(\ell_2\right) = \left(\left[v, \ell_1\right]\right)\left(\ell_2\right).$$

And we have $\left[u, \ell_1\right]_+ = \left[v, \ell_1\right]_+$

Proposition 3.1. *Let $u, v \in \mathcal{D}^m(L)$.*
If $u_+ = v_+$, then $u = v$.

Proof. Let $u, v \in \mathcal{D}^m(L)$ such that $u_+ = v_+$ and let $\alpha \in \mathbb{Z}^n$.

a) For $\alpha \in \mathbb{N}^n$, we have $u(x^\alpha) = v(x^\alpha)$ because $u_+ = v_+$.

b) Let $\alpha \in (\mathbb{Z}_-)^n$. We show by induction on the order m that $u(x^\alpha) = v(x^\alpha)$.

i) Let $m = 0$. As $u_+ = v_+$, so $u(1) = v(1)$. Yet $u(x^\alpha) = x^\alpha u(1)$ and $v(x^\alpha) = x^\alpha v(1)$, hence the equality $u(x^\alpha) = v(x^\alpha)$.

ii) Induction hypothesis 0 (**IH**0): Let $(s, m) \in \mathbb{N} \times \mathbb{N}^*$ such that $s \leq m$. If $u, v \in \mathcal{D}^s(L)$ such that $u_+ = v_+$, then $u(x^\alpha) = v(x^\alpha)$.
Let us show that if $u, v \in \mathcal{D}^{m+1}(L)$ such that $u_+ = v_+$, then
$u(x^\alpha) = v(x^\alpha)$.
Let $u, v \in \mathcal{D}^{m+1}(L)$ and $\ell \in L_+$. According to lemma 2, we have $[u, \ell]_+ = [v, \ell]_+$. Yet

$$[u, \ell], [v, \ell] \in \mathcal{D}^m(L).$$

So, $[u, \ell](x^\alpha) = [v, \ell](x^\alpha)$ according to (**IH**0). Taking $\ell = x^{-\alpha}$, we get

$$[u, x^{-\alpha}](x^\alpha) = [v, x^{-\alpha}](x^\alpha),$$

which means that

$$u(1) - x^{-\alpha} u(x^\alpha) = v(1) - x^{-\alpha} v(x^\alpha).$$

Since $u(1) = v(1)$, so

$$u(x^\alpha) = v(x^\alpha).$$

Consequently, if $u, v \in \mathcal{D}^m(L)$ such that $u_+ = v_+$, then

$$\forall \alpha \in (\mathbb{Z}_-)^n, u(x^\alpha) = v(x^\alpha).$$

c) Let $\alpha \notin (\mathbb{Z}_-)^n \cup \mathbb{N}^n$. In this case, there are $\alpha_1 \in (\mathbb{Z}_-)^n$ and $\alpha_2 \in (\mathbb{N})^n$ such that $\alpha = \alpha_1 + \alpha_2$. According to lemma 2, $[u, x^{\alpha_2}]_+ = [v, x^{\alpha_2}]_+$. Therefore, we deduce from *b*) that

$$[u, x^{\alpha_2}](x^{\alpha_1}) = [v, x^{\alpha_2}](x^{\alpha_1}).$$

In the same case, we have

$$u(x^{\alpha_1}) = v(x^{\alpha_1})$$

because $u_+ = v_+$. Yet

$$\begin{aligned} u(x^\alpha) &= u(x^{\alpha_1} x^{\alpha_2}) \\ &= [u, x^{\alpha_2}](x^{\alpha_1}) + x^{\alpha_2} u(x^{\alpha_1}), \end{aligned}$$

thus

$$\begin{aligned} u(x^\alpha) &= [v, x^{\alpha_2}](x^{\alpha_1}) + x^{\alpha_2} v(x^{\alpha_1}) \\ &= v(x^{\alpha_1} x^{\alpha_2}) = v(x^\alpha). \end{aligned}$$

So,

$$\forall \alpha \notin (\mathbb{Z}_-)^n \cup \mathbb{N}^n, \quad u(x^\alpha) = v(x^\alpha).$$

We conclude from these 3 cases, that for all $\ell \in L$, $u(\ell) = v(\ell)$, so $u = v$.

Proposition 3.2. *Let $m \in \mathbb{N}$.*
If $u \in \mathcal{D}^m(L)$ such that $u(V^m) \subseteq I$, then $u(L_+) \subseteq I$.

Proof. Let us remember that
$L_+ = k[x_1, .., x_n]$. Let $u \in \mathcal{D}^m(L)$ such that

$$u(V^m) \subseteq I \qquad (**).$$

Show by induction on the order m that for all $\ell \in L_+$, $u(\ell) \in I$.

1) $m = 0$.
$u \in \mathcal{D}^0(L)$ if and only if for all $\ell \in L_+$, $u(\ell) = \ell u(1_L)$. Yet $u(1_L) \in I$ (according to condition $(**)$) and I is an ideal, so for all $\ell \in L_+$, $u(\ell) \in I$.

2) Induction hypothesis 1 (**IH1**):
Let $s \in \mathbb{N}$ and $m \in \mathbb{N}^*$ such that $s < m$. If $u \in \mathcal{D}^s(L)$ such that $u(V^s) \subseteq I$, then $u(L_+) \subseteq I$.
Let us show that if $u \in \mathcal{D}^m(L)$ such that $u(V^m) \subseteq I$, then $u(L_+) \subseteq I$.
Suppose $u \in \mathcal{D}^m(L)$ such that $u(V^m) \subseteq I$ and $\alpha \in \mathbb{N}^n$. We prove that $u(x^\alpha) \in I$.

a) For $|\alpha| \leq m$, this is verified thanks to $(**)$.

b) Let $|\alpha| > m$. By induction on $|\alpha|$, we have:

i) Suppose $|\alpha| = m+1$. Then, there are $\alpha' \in \mathbb{N}^n$ with $|\alpha'| = m$ and $i \in \overline{1,n}$ such that $x^\alpha = x_i x^{\alpha'}$. By lemma 1, for all $b \in V^{m-1}$, $[u, x_i](b) \in I$. Yet

$$[u, x_i] \in \mathcal{D}^{m-1}(L).$$

Therefore, the hypotheses of (**IH1**) are verified for $[u, x_i]$, that is

$$[u, x_i] \in \mathcal{D}^{m-1}(L)$$

and

$$b \in V^{m-1}, [u, x_i](b) \in I.$$

By (**IH1**), we have

$$\forall \ell \in L_+, \quad [u, x_i](\ell) \in I.$$

In particular, $[u, x_i]\left(x^{\alpha'}\right) \in I$. Yet, $u\left(x^{\alpha'}\right) \in I$ because $|\alpha'| = m$ and we are in case 1, and I is an ideal. So

$$u(x^\alpha) = [u, x_i]\left(x^{\alpha'}\right) + x_i u\left(x^{\alpha'}\right) \in I.$$

ii) Induction hypothesis 2 (**IH2**):
Let $t, q \in \mathbb{N}$ such that $t > m+1$ and $q \geq m+1$.
If $q < t$ and $|\alpha| = q$, then $u(x^\alpha) \in I$.
Let us show that:
if $|\alpha| = t$ then $u(x^\alpha) \in I$.
Let $\alpha \in \mathbb{N}^n$ such that $|\alpha| = t$. In this case, there is $i \in \overline{1,n}$ and $\alpha' \in \mathbb{N}^n$ with $|\alpha'| = t-1$ such that $x^\alpha = x_i x^{\alpha'}$. By lemma 1, for all $b \in V^{m-1}$, $[u, x_i](b) \in I$. Yet $[u, x_i] \in \mathcal{D}^{m-1}(L)$. Therefore, the hypotheses of (**IH1**) are verified for $[u, x_i]$, that is

$$[u, x_i] \in \mathcal{D}^{m-1}(L) \text{ and }$$
$$b \in V^{m-1}, [u, x_i](b) \in I.$$

By (**IH1**), we have

$$\forall \ell \in L_+, \quad [u, x_i](\ell) \in I.$$

In particular $[u, x_i]\left(x^{\alpha'}\right) \in I$. Since $|\alpha'| < t$, then $u\left(x^{\alpha'}\right) \in I$ (*by* (**IH2**)). Hence

$$u(x^\alpha) = [u, x_i]\left(x^{\alpha'}\right) + x_i u\left(x^{\alpha'}\right).$$

So,

$$\forall \alpha \in \mathbb{N}^n, \quad u(x^\alpha) \in I.$$

Therefore,

$$\forall \ell \in L_+, \; u(\ell) \in I.$$

It follows that for $s = m$, if $u \in \mathcal{D}^m(L)$ such that $u(V^m) \in I$, then for all $\ell \in L_+$, $u(\ell) \in I$.

Consequently, for $m \in \mathbb{N}$, if $u \in \mathcal{D}^m(L)$ such that $u(V^m) \in I$, then $u(L_+) \subseteq I$.

Corollary 3.1. *Let $m \in \mathbb{N}$.*
If $u \in \mathcal{D}^m(L)$ such that $u(V^m) \subseteq (0)$, then $u_+ = 0_{End(L_+)}$.

Proof. By setting $I = (0)$, we deduce from Proposition 3.2 that for all $\ell \in L_+$, $u(\ell) = 0$. It follows that $u_+ = 0_{End(L_+)}$.

Corollary 3.2. *Let $m \in \mathbb{N}$.*
If $u, v \in \mathcal{D}^m(L)$ such that for any $a \in V^m$, $u(a) = v(a)$, then $u_+ = v_+$.

Proof. By setting $d = u - v$, we get that $d \in \mathcal{D}^m(L)$ and

$$\forall a \in V^m, \quad d(a) \in (0).$$

By Corollary 3.1, we have $d_+ = 0_{End(L)}$ and $u_+ = v_+$.

Corollary 3.3. *Let $m \in \mathbb{N}$ and $u, v \in \mathcal{D}^m(L)$.*
If for all $a \in V^m$, $u(a) = v(a)$, then $u = v$.

Proof. Suppose that

$$\forall a \in V^m, \quad u(a) = v(a).$$

According to the Corollary 3.2, $u_+ = v_+$. Therefore, we deduce from the Proposition 3.1 that $u = v$.

This "sufficient condition" (cf. Corollary 3.3) is the key that will allow us to determine exactly the algebra of the differential operators on the Laurent ring.

3.2 k is a zero characteristic field

Thanks to the sufficient condition obtained above, we show that, contreatment to the ring of polynomials, the algebra of differential operators on Laurent polynomials ring contains a Weyl algebra as indicated by the following theorem.

Theorem 3.4. *Let k be a zero characteristic field. The algebra of differential operators on a Laurent polynomials ring L is*

$$\begin{aligned}\mathcal{D}(L) &= k\left[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, \partial_1, \ldots, \partial_n\right] \\ &= L\left[\partial_1, \ldots, \partial_n\right].\end{aligned}$$

Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and $|\alpha| = \sum_{s=1}^{n} \alpha_i$.

1) Let us show that

$$k\left[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, \partial_1, \ldots, \partial_n\right] \subset \mathcal{D}(L).$$

By Proposition 2.2,

$$\mathcal{D}^1(L) = L \oplus Der_k(L),$$

so for all $i \in \overline{1,n}$, $\partial_i \in \mathcal{D}^1(L)$. Yet for all $f \in \mathcal{D}^p(L)$ and $g \in \mathcal{D}^q(L)$, $f \circ g \in \mathcal{D}^{p+q}(L)$, so any element h of $L[\partial_1, \ldots, \partial_n]$ of the form $\sum_{|\alpha|\leq m} \ell_\alpha \partial^\alpha$, is a differential operator on L of order m. Therefore, any element of $L[\partial_1, \ldots, \partial_n]$ is a differential operator on L. Hence,

$$L[\partial_1, \ldots, \partial_n] \subseteq \mathcal{D}(L).$$

2) Let us show that $\mathcal{D}(L) \subseteq L[\partial_1, \ldots, \partial_n]$. The method consists in showing that, for all $d \in \mathcal{D}^m(L)$, we can construct $d' \in L[\partial_1, \ldots, \partial_n]$ such that for all $\ell \in L$, $d'(\ell) = d(\ell)$. Let $m \in \mathbb{N}$, $d \in \mathcal{D}^m(R)$ and let us show that $d \in L[\partial_1, \ldots, \partial_n]$. Suppose $d' = \sum_{|\alpha|\leq m} \ell_\alpha \partial^\alpha$ such that

$$\forall a \in V^m, \quad d'(a) = d(a).$$

This equality leads to a triangular system which allow to express the unknowns ℓ_α according to $d(x^\alpha)$ where $|\alpha| \leq m$. Thus, we have constructed $d' \in \mathcal{D}^m(L)$ such that

$$\forall a \in V^m, \quad d'(a) = d(a).$$

We deduce from Corollary 3.3 that $d = d'$ and it follows that

$$d = \sum_{|\alpha|\leq m} \ell_\alpha \partial^\alpha \in L[\partial_1, \ldots, \partial_n],$$

hence

$$\mathcal{D}^m(L) \subset L[\partial_1, \ldots, \partial_n].$$

According to 1) and 2), we have

$$\mathcal{D}(L) = L[\partial_1, \ldots, \partial_n].$$

Remark 3.1. *Let $n \in \mathbb{N}^*$.*

$$\mathcal{D}(k[x_1, \ldots, x_n]) = A_n(k) \subseteq \mathcal{D}(L).$$

3.3 k is a nonzero characteristic field.

k is a field of characteristic $p > 0$.

Definition 3.1. *Let $m \in \mathbb{N}$, $i \in \overline{1,n}$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$. We define the following map:*

$$\begin{aligned} d_i^{[m]} : L &\longrightarrow L \\ \sum_{\alpha\in\mathbb{Z}^n} a_\alpha x^\alpha &\longmapsto \sum_{\alpha\in\mathbb{Z}^n} a_\alpha \frac{\partial_i^m (x^\alpha)}{m!}.\end{aligned}$$

Proposition 3.3. *Let $i \in \overline{1,n}$. For all $m \in \mathbb{N}$, $d_i^{[m]} \in \mathcal{D}^m(L)$.*

Proof.

1) As $d_i^{[m]} = \dfrac{\partial_i^m}{m!}$ and $\partial_i^m \in End_k(L)$, so $d_i^{[m]} \in End_k(L)$.

2) Let $\ell \in L$. We have

$$\left[d_i^{[m]}, \ell\right] = \left[\frac{\partial_i^m}{m!}\ell\right] = \frac{1}{m!}\left[\partial_i^m, \ell\right].$$

Yet $[\partial_i^m, \ell] \in \mathcal{D}^{m-1}(L)$ because $\partial_i^m \in \mathcal{D}^m(L)$, so

$$\forall \ell \in L, \left[d_i^{[m]}, \ell\right] \in \mathcal{D}^{m-1}(L).$$

It follows that $d_i^{[m]} \in \mathcal{D}^m(L)$.

Theorem 3.5. *Let $m \in \mathbb{N}$. The algebra of differential operators on Laurent polynomials ring L is*

$$\begin{aligned}\mathcal{D}(L) &= k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, \ldots, d_i^{[m]}, \ldots] \\ &= L\left[\ldots, d_i^{[m]}, \ldots\right], \forall i = 1, \ldots, n.\end{aligned}$$

Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and $|\alpha| = \sum_{s=1}^{n} \alpha_i$.

1) Let us show that

$$L\left[\ldots, d_i^{[m]}, \ldots\right] \subseteq \mathcal{D}(L).$$

Let

$$h = \sum_{|\alpha| \leq m} \ell_\alpha d^{[\alpha]} \in L\left[\ldots, d_i^{[m]}, \ldots\right].$$

According to Proposition 3.3, $d_i^{[m]} \in \mathcal{D}^m(L)$, so for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| = m$, we obtain $d^{[\alpha]} \in \mathcal{D}^m(L)$, therefore $h \in \mathcal{D}^m(L)$. It follows that

$$L\left[\ldots, d_i^{[m]}, \ldots\right] \subseteq \mathcal{D}(L).$$

2) Let us show that

$$\mathcal{D}(L) \subseteq L\left[\ldots, d_i^{[m]}, \ldots\right].$$

Let $m \in \mathbb{N}$, $h \in \mathcal{D}^m(L)$ and $h' = \sum_{|\alpha| \leq m} \ell_\alpha d^{[\alpha]}$ such that

$$\forall v \in V^m, \quad h'(v) = h(v).$$

This condition leads to a triangular system which allow to express the unknowns l_α according to $h(x^\alpha)$ where $|\alpha| \leq m$.
Thus, we have constructed $h' \in \mathcal{D}^m(L)$ such that

$$\forall a \in V^m, \quad d'(a) = d(a).$$

We deduce from Corollary 3.3 that $h = h'$. It follows that

$$h \in L\left[\ldots, d_i^{[m]}, \ldots\right],$$

hence

$$\mathcal{D}^m(L) \subseteq \mathcal{D}(L) \subseteq L\left[\ldots, d_i^m, \ldots\right].$$

By 1) and 2), we have

$$\mathcal{D}(L) = L\left[\ldots, d_i^m, \ldots\right].$$

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Appendix

Notation 1. *Let* $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ *and* $|\alpha| = \sum_{i}^{n} \alpha_i$. *We denote by:*

1) $x = (x_1, \ldots, x_n)$.
2) $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$.
3) *For all* $i = 1, \ldots, n$, $\partial_i = \dfrac{\partial}{\partial_{x_i}}$ *and*

$$\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}.$$

4) $d^{[\alpha]} = d_1^{[\alpha_1]} \ldots d_n^{[\alpha_n]}$.
5) *For all* $m \in \mathbb{N}$, V^m *the vector space generated by all the* x^α *with* $|\alpha| \leq m$.
6) $L_+ = k[x_1, \ldots, x_n]$.
7) *For all* $u \in End_k(L)$, u_+ *the restriction of* u *on* L_+.