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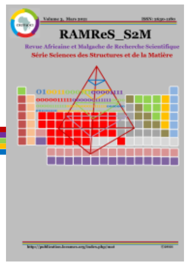

q-Bernstein basis functions on a triangulated domain

Soro Sionfon Simon[1,*], Haudié Jean Stéphane Inkpé[2], Koua Brou Jean Claude[3]

[1]*Université Felix Houphouët Boigny*

[2]*Digital Research and Expertise Unit, Université Virtuelle de Côte d'Ivoire (UVCI), 28 BP 536 Abidjan*

[3]*Department of Mathematics, Université Felix Houphouët Boigny*

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***Abstract*:**

In this paper, we propose generalized q-Bernstein basis functions based on a triangulated domain using q-real. Proofs of some geometric and algebraic properties are given. These include the partition of unity, degree raising and the q-De Casteljau algorithm. Finally, we introduce the generalized q-Bernstein polynomial functions of degree n. We have constructed these polynomials on the triangle with the parameter q, which can be used to improve the rendering given by standard Bernstein polynomials.

***Keywords*:** q-Bernstein basis; q-Integers; q-Bernstein polynomials.

MSC (2020): 65D07, 65D17, 65D10.

1 Introduction and definitions

Bernstein polynomial bases are polynomial functions used as basic functions in the construction and handling of Bezier curves and surfaces. Bernstein polynomial bases play an important role in shape conservation of curves and surfaces [1][2] in computer-aided geometric design (CAGD), but also in geometric modelling and approximation theory.

While classical Bernstein polynomials were introduced by Bernstein in 1912 [3], classical Bezier curves and surfaces were developed in the 1960s by the well-known French engineer Bezier, who worked for the car manufacturer Renault. Bezier used these curves and surfaces to model and manufacture vehicles with aerodynamic bodies [4]. The CAGD emerged from the patch theory of rectangular surfaces by Coons (from MIT) [5] and Bezier (from Re-

*Corresponding author:

Email address: sorosionfon@yahoo.fr (S.*S. Soro*)

nault) in the late 1960s.
Since it is not easy, in practice, to exactly reproduce Bezier curves and surfaces from classical Bezier base polynomials, it has been introduced in the literature, the univariate h-Bernstein polynomials (Goldman 1985; Goldman and Barry 1991) [6][7] and the univariate q-Bernstein polynomials (Phillips et al. 1997) [8] which accept a tolerance h close to zero or q close to one. The first q-analogous form of the (rational) Bernstein operators was studied by Lupas [9] due to the development of quantum computing [10]. The extension to the bivariate case was considered in [11] for the classical form (without tolerance) on the triangle and in [12][13] for the quadrilateral case.
In 2020, Lamberti et al [14] construct an h-Bernstein basis on a triangle. In 2021, Khan [15] constructs new operators based on the Phillips quantum analog. These Bernstein-type operators also interpolate the value of a given function on the C1 boundary of the triangle.
In view of the above remarks, this paper focuses on the definition of new q-Bernstein basis functions of degree n on a triangulated domain and on the study of several of their properties, namely, the recurrence relation, the partition of the unit, the degree raising, the linear independence (basis of the space of polynomials of total degree less than or equal to n) on a triangle. We also propose the q-de Casteljau algorithm which is a recursive evaluation algorithm. We conclude this paper with graphical examples of q-Bernstein Bezier patches on a triangular domain, emphasizing the effect of changing the q-parameter.

2 q-Bernstein basis polynomial functions on a triangle

Let $V := (x, y)$ be a point of nondegenerate triangle $T := (V_1, V_2, V_3)$ of $\mathbb{R}^2$ where $V_i := (x_i, y_i)$, $i = 1, 2, 3$ are the vertices of T. The barycentric coordinates (τ_1, τ_2, τ_3) verify the equation:

$$V = \sum_{i=1}^{3} \tau_i V_i, \quad \text{with} \quad \sum_{i=1}^{3} \tau_i = 1,$$

we show that this equation has one solution (Cramer's rule):

$$\tau_1 = \frac{\Delta_1}{\delta}, \qquad \tau_2 = \frac{\Delta_2}{\delta}, \qquad \tau_3 = \frac{\Delta_3}{\delta}$$

with

$$\delta := \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}, \quad \Delta_1 := \begin{vmatrix} x & x_2 & x_3 \\ y & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix},$$

$$\Delta_2 := \begin{vmatrix} x_1 & x & x_3 \\ y_1 & y & y_3 \\ 1 & 1 & 1 \end{vmatrix}, \quad \Delta_3 := \begin{vmatrix} x_1 & x_2 & x \\ y_1 & y_2 & y \\ 1 & 1 & 1 \end{vmatrix}.$$

Let $n \in \mathbb{N}^*$, $\lambda := (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{N}^3$

Consider: $|\lambda| = \lambda_1 + \lambda_2 + \lambda_3$,
$\epsilon^1 := (1, 0, 0)$, $\epsilon^2 := (0, 1, 0)$, $\epsilon^3 := (0, 0, 1)$.

Let $q \in \mathbb{R}$. For any integer $n = 1, 2, \cdots$, we have:

$[n]_q$ is defined [10]:
$[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$, $\quad [0]_q = 1$

the q-factorial is defined by [10]:
$[n]_q! = [1]_q [2]_q \cdots [n]_q$, $\quad [0]_q! = 1$

and for $q \neq -1$ the q-binomial coefficient for $|\lambda| = n$ is given by:

$$\begin{bmatrix} n \\ \lambda \end{bmatrix}_q := \frac{[n]_q!}{[\lambda_1]_q! [\lambda_2]_q! [\lambda_3]_q!}$$

Lemma 2.1 .
The q-binomial coefficients satisfy the recur-

rence relation :

$$\begin{bmatrix} n \\ \lambda \end{bmatrix}_q = \begin{bmatrix} n-1 \\ \lambda - \epsilon^1 \end{bmatrix}_q + q^{\lambda_1} \begin{bmatrix} n-1 \\ \lambda - \epsilon^2 \end{bmatrix}_q + q^{\lambda_1+\lambda_2} \begin{bmatrix} n-1 \\ \lambda - \epsilon^3 \end{bmatrix}_q \quad (1)$$

Proof

$$\begin{aligned}
[n]_q =& 1 + q + q^2 + \cdots + q^{n-1} \\
=& \left(1 + q + \cdots + q^{\lambda_1 - 1}\right) + \\
& q^{\lambda_1}\left(1 + q + \cdots + q^{n-\lambda_1-1}\right) \\
=& [\lambda_1]_q + q^{\lambda_1}\Big[1 + q + \cdots + q^{\lambda_2-1} + \\
& q^{\lambda_2}\left(1 + q + \cdots + q^{n-\lambda_1-\lambda_2-1}\right)\Big] \\
=& [\lambda_1]_q + q^{\lambda_1}\left(1 + q + \cdots + q^{\lambda_2-1}\right) + \\
& q^{\lambda_1}q^{\lambda_2}\left(1 + q + \cdots + q^{n-\lambda_1-\lambda_2-1}\right) \\
=& [\lambda_1]_q + q^{\lambda_1}[\lambda_2]_q + \\
& q^{\lambda_1}q^{\lambda_2}\left(1 + q + \cdots + q^{\lambda_3-1}\right) \\
=& [\lambda_1]_q + q^{\lambda_1}[\lambda_2]_q + q^{\lambda_1}q^{\lambda_2}[\lambda_3]_q
\end{aligned}$$

Using this last expression of the above equality, we obtain

$$\begin{aligned}
\begin{bmatrix} n \\ \lambda \end{bmatrix}_q =& \frac{[n]_q!}{[\lambda_1]_q![\lambda_2]_q![\lambda_3]_q!} \\
=& \frac{[n-1]_q![n]_q}{[\lambda_1]_q![\lambda_2]_q![\lambda_3]_q!} \\
=& \frac{[n-1]_q!\left([\lambda_1]_q + q^{\lambda_1}[\lambda_2]_q + q^{\lambda_1}q^{\lambda_2}[\lambda_3]_q\right)}{[\lambda_1]_q![\lambda_2]_q![\lambda_3]_q!} \\
=& \frac{[n-1]_q![\lambda_1]_q}{[\lambda_1]_q![\lambda_2]_q![\lambda_3]_q!} + q^{\lambda_1}\frac{[n-1]_q![\lambda_2]_q}{[\lambda_1]_q![\lambda_2]_q![\lambda_3]_q!} + \\
& q^{\lambda_1}q^{\lambda_2}\frac{[n-1]_q![\lambda_3]_q}{[\lambda_1]_q![\lambda_2]_q![\lambda_3]_q!} \\
=& \frac{[n-1]_q!}{[\lambda_1-1]_q![\lambda_2]_q![\lambda_3]_q!} + \\
& q^{\lambda_1}\frac{[n-1]_q!}{[\lambda_1]_q![\lambda_2-1]_q![\lambda_3]_q!} + \\
& q^{\lambda_1}q^{\lambda_2}\frac{[n-1]_q!}{[\lambda_1]_q![\lambda_2]_q![\lambda_3-1]_q!} \\
=& \begin{bmatrix} n-1 \\ \lambda - \epsilon^1 \end{bmatrix}_q + q^{\lambda_1}\begin{bmatrix} n-1 \\ \lambda - \epsilon^2 \end{bmatrix}_q + \\
& q^{\lambda_1+\lambda_2}\begin{bmatrix} n-1 \\ \lambda - \epsilon^3 \end{bmatrix}_q
\end{aligned}$$

□

For a real number $q \neq -1$ and the triplet λ above, we define the polynomial functions q-Bernstein on the triangle T as follows:

$$B^n_\lambda(\tau; T, q) := \begin{cases} \bullet \begin{bmatrix} n \\ \lambda \end{bmatrix}_q \prod_{k=0}^{\lambda_1-1}\left(1-(1-\tau_1)q^k\right) \times \\ \quad \prod_{k=0}^{\lambda_2-1}\left(1-\tau_1-\tau_3 q^k\right) \times \prod_{k=0}^{\lambda_3-1}\tau_3 \\ \text{for } n \geq 1 \quad \text{and } |\lambda| = n \\ \bullet \quad 1, \qquad \text{for } n = 0 \quad \text{and } |\lambda| = 0 \end{cases} \quad (2)$$

In 2, for $i = 1, 2, 3$ if $\lambda_i = 0$ then $\prod_{k=0}^{\lambda_i} ... = 1$ and for $\lambda_3 \geq 1$, $\prod_{k=0}^{\lambda_3 - 1} \tau_3 = \tau_3^{\lambda_3}$

We note that $B_\lambda^n(\tau; T, q) \geq 0$ for $-1 < q \leq 1$. Because barycentric coordinates are nonnegative on the triangle [2] and their sum is equal to 1.
We also note that if $q = 1$, $B_\lambda^n(\tau; T, 1)$ are the polynomials of Bernstein's natural basis.(see [11])

For an arbitrary interval $[a, b]$ with $(a, b) \in \mathbb{R}^2$, the q-Bernstein polynomial functions have been defined by the following set [16]:

$$B_k^n(t, [a,b], q) = \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\prod_{i=0}^{k-1}\Big(t - a + a(1-q^i)\Big)\prod_{i=0}^{n-k-1}\Big(b - t + t(1-q^i)\Big)}{\prod_{i=0}^{n-1}\Big(b - a + a(1-q^i)\Big)}, \quad k = 0, \cdots, n \tag{3}$$

with: $b - a + a(1 - q^i) \neq 0$ for some q for which $1 \leq i \leq n+1$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$

Throughout the following, unless otherwise stated $[\cdots] = [\cdots]_q$

3 Recurrence relations for the q-Bernstein polynomial functions

From the expression of $B_\lambda^n(\tau; T, q)$ in 2 we deduce that:

$$B_\lambda^n(\tau; T, q) := \Big(1 - (1-\tau_1)q^{\lambda_1}\Big) B_{\lambda-\epsilon^1}^{n-1}(\tau; T, q) + q^{\lambda_1}\Big(1 - \tau_1 - \tau_3 q^{\lambda_2}\Big) B_{\lambda-\epsilon^2}^{n-1}(\tau; T, q) + q^{\lambda_1+\lambda_2}\tau_3 B_{\lambda-\epsilon^3}^{n-1}(\tau; T, q) \tag{4}$$

Proof.

$$\Big(1 - (1-\tau_1)q^{\lambda_1}\Big) B_{\lambda-\epsilon^1}^{n-1}(\tau; T, q) + q^{\lambda_1}\Big(1 - \tau_1 - \tau_3 q^{\lambda_2}\Big) B_{\lambda-\epsilon^2}^{n-1}(\tau; T, q) + q^{\lambda_1+\lambda_2}\tau_3 B_{\lambda-\epsilon^3}^{n-1}(\tau; T, q) =$$

$$\Big(1 - (1-\tau_1)q^{\lambda_1}\Big) \begin{bmatrix} n-1 \\ \lambda - \epsilon^1 \end{bmatrix}_q \times \prod_{k=0}^{\lambda_1-1-1}\Big(1 - (1-\tau_1)q^k\Big) \prod_{k=0}^{\lambda_2-1}\Big(1 - \tau_1 - \tau_3 q^k\Big) \times \prod_{k=0}^{\lambda_3-1}\tau_3 + q^{\lambda_1}\Big(1 - \tau_1 - \tau_3 q^{\lambda_2}\Big) \begin{bmatrix} n-1 \\ \lambda - \epsilon^2 \end{bmatrix}_q \times \prod_{k=0}^{\lambda_1-1}\Big(1 - (1-\tau_1)q^k\Big) \prod_{k=0}^{\lambda_2-1-1}\Big(1 - \tau_1 - \tau_3 q^k\Big) \times \prod_{k=0}^{\lambda_3-1}\tau_3 + q^{\lambda_1+\lambda_2}\tau_3 \begin{bmatrix} n-1 \\ \lambda - \epsilon^3 \end{bmatrix}_q \times \prod_{k=0}^{\lambda_1-1}\Big(1 - (1-\tau_1)q^k\Big) \prod_{k=0}^{\lambda_2-1}\Big(1 - \tau_1 - \tau_3 q^k\Big) \times \prod_{k=0}^{\lambda_3-1-1}\tau_3$$

$$= \begin{bmatrix} n-1 \\ \lambda - \epsilon^1 \end{bmatrix}_q \prod_{k=0}^{\lambda_1-1}\Big(1 - (1-\tau_1)q^k\Big) \times \prod_{k=0}^{\lambda_2-1}\Big(1 - \tau_1 - \tau_3 q^k\Big) \prod_{k=0}^{\lambda_3-1}\tau_3 + q^{\lambda_1} \begin{bmatrix} n-1 \\ \lambda - \epsilon^2 \end{bmatrix}_q \prod_{k=0}^{\lambda_1-1}\Big(1 - (1-\tau_1)q^k\Big) \times$$

$$\prod_{k=0}^{\lambda_2-1}\left(1-\tau_1-\tau_3 q^k\right)\prod_{k=0}^{\lambda_3-1}\tau_3+$$

$$q^{\lambda_1+\lambda_2}\begin{bmatrix} n-1 \\ \lambda-\epsilon^3 \end{bmatrix}_q \prod_{k=0}^{\lambda_1-1}\left(1-(1-\tau_1)q^k\right)\times$$

$$\prod_{k=0}^{\lambda_2-1}\left(1-\tau_1-\tau_3 q^k\right)\prod_{k=0}^{\lambda_3-1}\tau_3$$

$$=\left(\begin{bmatrix} n-1 \\ \lambda-\epsilon^1 \end{bmatrix}_q + q^{\lambda_1}\begin{bmatrix} n-1 \\ \lambda-\epsilon^2 \end{bmatrix}_q + \right.$$

$$\left. q^{\lambda_1+\lambda_2}\begin{bmatrix} n-1 \\ \lambda-\epsilon^3 \end{bmatrix}_q \right)\prod_{k=0}^{\lambda_1-1}\left(1-(1-\tau_1)q^k\right)\times$$

$$\prod_{k=0}^{\lambda_2-1}\left(1-\tau_1-\tau_3 q^k\right)\prod_{k=0}^{\lambda_3-1}\tau_3$$

$$=\begin{bmatrix} n \\ \lambda \end{bmatrix}_q \prod_{k=0}^{\lambda_1-1}\left(1-(1-\tau_1)q^k\right)\times$$

$$\prod_{k=0}^{\lambda_2-1}\left(1-\tau_1-\tau_3 q^k\right)\prod_{k=0}^{\lambda_3-1}\tau_3 \quad \text{by applying of 1}$$

$$=B_\lambda^n(\tau;T,q)$$

4 Partition of unity

Proposition 1 *The q-Bernstein polynomial functions on a triangle verify the partition of the unit.*

$$\sum_{|\lambda|=n} B_\lambda^n(\tau;T,q)=1 \tag{5}$$

Proof.

For $n=0$: $B_0^0(\tau;T,q)=1$

we assume that relation 5 is true at rank $n \geq 0$

$$\sum_{|\lambda'|=n+1} B_{\lambda'}^{n+1}(\tau;T,q)=$$

$$\sum_{|\lambda'|=n+1}\left[\left(1-(1-\tau_1)q^{\lambda'_1}\right)B^n_{\lambda'-\epsilon^1}(\tau;T,q)+\right.$$

$$q^{\lambda'_1}\left(1-\tau_1-\tau_3 q^{\lambda'_2}\right)B^n_{\lambda'-\epsilon^2}(\tau;T,q)+$$

$$\left.q^{\lambda'_1+\lambda'_2}\tau_3 B^n_{\lambda'-\epsilon^3}(\tau;T,q)\right]$$

$$=\sum_{|\lambda'|=n+1}\left(1-(1-\tau_1)q^{\lambda'_1}\right)B^n_{\lambda'-\epsilon^1}(\tau;T,q)+$$

$$\sum_{|\lambda'|=n+1} q^{\lambda'_1}\left(1-\tau_1-\tau_3 q^{\lambda'_2}\right)B^n_{\lambda'-\epsilon^2}(\tau;T,q)+$$

$$\sum_{|\lambda'|=n+1} q^{\lambda'_1+\lambda'_2}\tau_3 B^n_{\lambda'-\epsilon^3}(\tau;T,q)$$

$$=\sum_{|\lambda|=n}\left(1-(1-\tau_1)q^{\lambda'_1}\right)B^n_{\lambda}(\tau;T,q)+$$

$$\sum_{|\lambda|=n} q^{\lambda'_1}\left(1-\tau_1-\tau_3 q^{\lambda'_2}\right)B^n_{\lambda}(\tau;T,q)+$$

$$\sum_{|\lambda|=n} q^{\lambda'_1+\lambda'_2}\tau_3 B^n_{\lambda}(\tau;T,q)$$

$$=\sum_{|\lambda|=n}\left[\left(1-(1-\tau_1)q^{\lambda'_1}\right)+\right.$$

$$q^{\lambda'_1}\left(1-\tau_1-\tau_3 q^{\lambda'_2}\right)+$$

$$\left.q^{\lambda'_1+\lambda'_2}\tau_3\right]B^n_{\lambda}(\tau;T,q)$$

$$=\sum_{|\lambda|=n} B_\lambda^n(\tau;T,q)=1$$

5 Degree elevation for q-Bernstein polynomials

Proposition 2 *The q-Bernstein Bezier polynomial $b^n(\tau;T,q)$ of degree n, with*

$$b^n(\tau;T,q)=\sum_{|\lambda|=n} b_\lambda B_\lambda^n(\tau;T,q) \tag{6}$$

can be written as follows:

$$b^n(\tau;T,q)=\sum_{|\mu|=n+1} b_\mu^{(1)} B_\mu^{n+1}(\tau;T,q) \tag{7}$$

where

$$b_{\mu}^{(1)} = \frac{1}{[n+1]} \sum_{k=1}^{3} \mu_k b_{\mu-\epsilon^k}, \qquad |\mu| = n+1.$$

Proof.

$$b^n(\tau; T, q) = \sum_{|\lambda|=n} b_\lambda B_\lambda^n(\tau; T, q) \tag{8}$$

is the q-Bernstein Bezier polynomial of degree n. The product of $B_\lambda^n(\tau; T, q)$ by

$$1 = \left(1-(1-\tau_1)q^{\lambda_1}\right) + q^{\lambda_1}\left(1-\tau_1-\tau_3 q^{\lambda_2}\right) + q^{\lambda_1+\lambda_2}\tau_3$$

gives

$$b^n(\tau; T, q) = \sum_{|\lambda|=n} b_\lambda B_\lambda^n(\tau; T, q) \times \left[\left(1-(1-\tau_1)q^{\lambda_1}\right) + q^{\lambda_1}\left(1-\tau_1-\tau_3 q^{\lambda_2}\right) + q^{\lambda_1+\lambda_2}\tau_3\right]$$

By development of $B_\lambda^n(\tau; T, q)$, we show that

$$\begin{aligned} b^n(\tau; T, q) = {} & \sum_{|\lambda|=n} b_\lambda \begin{bmatrix} n \\ \lambda \end{bmatrix}_q \prod_{k=0}^{\lambda_1} \left(1-(1-\tau_1)q^k\right) \times \prod_{k=0}^{\lambda_2-1} \left(1-\tau_1-\tau_3 q^k\right) \prod_{k=0}^{\lambda_3-1} \tau_3 + {} \\ & \sum_{|\lambda|=n} q^{\lambda_1} b_\lambda \begin{bmatrix} n \\ \lambda \end{bmatrix}_q \prod_{k=0}^{\lambda_1-1} \left(1-(1-\tau_1)q^k\right) \times \prod_{k=0}^{\lambda_2} \left(1-\tau_1-\tau_3 q^k\right) \prod_{k=0}^{\lambda_3-1} \tau_3 + {} \\ & \sum_{|\lambda|=n} q^{\lambda_1+\lambda_2} b_\lambda \begin{bmatrix} n \\ \lambda \end{bmatrix}_q \prod_{k=0}^{\lambda_1-1} \left(1-(1-\tau_1)q^k\right) \times \prod_{k=0}^{\lambda_2-1} \left(1-\tau_1-\tau_3 q^k\right) \prod_{k=0}^{\lambda_3} \tau_3 \end{aligned} \tag{9}$$

$\forall\ \lambda' = (\lambda'_1, \lambda'_2, \lambda'_3)$ such that $|\lambda'| = n+1$, the term

$$\begin{bmatrix} n+1 \\ \lambda' \end{bmatrix}_q \prod_{k=0}^{\lambda'_1-1} \left(1-(1-\tau_1)q^k\right) \times \prod_{k=0}^{\lambda'_2-1} \left(1-\tau_1-\tau_3 q^k\right) \prod_{k=0}^{\lambda'_3-1} \tau_3$$

appears in the first, second and third terms of 9 respectively with the coefficients $\frac{[\lambda'_1]}{[n+1]} b_{\lambda'-\epsilon^1}$, $q^{\lambda_1}\frac{[\lambda'_2]}{[n+1]} b_{\lambda'-\epsilon^2}$, $q^{\lambda_1+\lambda_2}\frac{[\lambda'_3]}{[n+1]} b_{\lambda'-\epsilon^3}$.
These coefficients sum to

$$b_{\lambda'}^{(1)} = \frac{1}{[n+1]}\Big([\lambda'_1] b_{\lambda'-\epsilon^1} + q^{\lambda_1}[\lambda'_2] b_{\lambda'-\epsilon^2} + q^{\lambda_1+\lambda_2}[\lambda'_3] b_{\lambda'-\epsilon^3}\Big).$$

$\square$

We deduce the following:

Corollary 1

$$x = \sum_{|\lambda|=n} \frac{1}{[n]}\Big([\lambda_1]x_1 + [\lambda_2]x_2 + [\lambda_3]x_3\Big)\times B_\lambda^n(\tau;T,q) \quad (10)$$

$$y = \sum_{|\lambda|=n} \frac{1}{[n]}\Big([\lambda_1]y_1 + [\lambda_2]y_2 + [\lambda_3]y_3\Big)\times B_\lambda^n(\tau;T,q) \quad (11)$$

Proof.(by recurrence)

For $n=1$ we have

$$\begin{aligned} x =& \tau_1x_1 + \tau_2x_2 + \tau_3x_3 \\ =& \sum_{|\lambda|=1}\Big([\lambda_1]x_1 + [\lambda_2]x_2 + [\lambda_3]x_3\Big)B_\lambda^1(\tau;T,q). \end{aligned}$$

because

$$\begin{aligned} &[\lambda_1]x_1B_\lambda^1(\tau;T,q) = \\ &[\lambda_1]x_1 \begin{bmatrix} 1 \\ \lambda \end{bmatrix}_q \prod_{k=0}^{\lambda_1-1}\Big(1-(1-\tau_1)q^k\Big)\times \\ &\prod_{k=0}^{\lambda_2-1}\Big(1-\tau_1-\tau_3q^k\Big)\prod_{k=0}^{\lambda_3-1}\tau_3 \\ &= \tau_1x_1 \quad \text{for n=1 and by taking} \quad \lambda_1 = 1. \end{aligned}$$

Similarly, we obtain:

- $[\lambda_2]x_2B_\lambda^1(\tau;T,q) = \tau_2x_2$
  for n=1 and by taking $\lambda_2 = 1$.

- $[\lambda_3]x_3B_\lambda^1(\tau;T,q) = \tau_3x_3$
  for n=1 and by taking $\lambda_3 = 1$.

Assume equality for 10 is true $n \geq 1$ then we have:

$$x = \sum_{|\lambda|=n} \frac{1}{[n]}\Big([\lambda_1]x_1+[\lambda_2]x_2+[\lambda_3]x_3\Big)B_\lambda^n(\tau;T,q)$$

Let

$$c_\lambda := \frac{1}{[n]}\Big([\lambda_1]x_1 + [\lambda_2]x_2 + [\lambda_3]x_3\Big), \quad |\lambda| = n$$

be the coefficients in the expression for x.

By formula 7, we deduce that

$$x = \sum_{|\lambda'|=n+1} c_{\lambda'}^{(1)}B_{\lambda'}^{n+1}(\tau;T,q)$$

with

$$c_{\lambda'}^{(1)} = \frac{1}{[n+1]}\Big([\lambda_1']c_{\lambda'-\epsilon^1} + q^{\lambda_1}[\lambda_2']c_{\lambda'-\epsilon^2} + q^{\lambda_1+\lambda_2}[\lambda_3']c_{\lambda'-\epsilon^3}\Big) \ ; |\lambda'| = n+1.$$

Development of the coefficient $c_{\lambda'}^{(1)}$ gives

$$\begin{aligned} &[n+1]c_{\lambda'}^{(1)} = \\ &\frac{[\lambda'_1]}{[n]}\Big([\lambda'_1-1]x_1 + [\lambda'_2]x_2 + [\lambda'_3]x_3\Big)+ \\ &q^{\lambda_1}\frac{[\lambda'_2]}{[n]}\Big([\lambda'_1]x_1 + [\lambda'_2-1]x_2 + [\lambda'_3]x_3\Big)+ \\ &q^{\lambda_1+\lambda_2}\frac{[\lambda'_3]}{[n]}\Big([\lambda'_1]x_1 + [\lambda'_2]x_2 + [\lambda'_3-1]x_3\Big) \\ =&\frac{[\lambda'_1]}{[n]}\Big([\lambda_1]x_1 + [\lambda_2]x_2 + [\lambda_3]x_3\Big)+ \end{aligned}$$

$$q^{\lambda_1}\frac{[\lambda'_2]}{[n]}\Big([\lambda_1]x_1+[\lambda_2]x_2+[\lambda_3]x_3\Big)+$$
$$q^{\lambda_1+\lambda_2}\frac{[\lambda'_3]}{[n]}\Big([\lambda_1]x_1+[\lambda_2]x_2+[\lambda_3]x_3\Big)$$
$$=\frac{[\lambda'_1]x_1}{[n]}\Big([\lambda_1]+q^{\lambda_1}[\lambda_2]+q^{\lambda_1+\lambda_2}[\lambda_3]\Big)+$$
$$\frac{[\lambda'_2]x_2}{[n]}\Big([\lambda_1]+q^{\lambda_1}[\lambda_2]+q^{\lambda_1+\lambda_2}[\lambda_3]\Big)+$$
$$\frac{[\lambda'_3]x_3}{[n]}\Big([\lambda_1]+q^{\lambda_1}[\lambda_2]+q^{\lambda_1+\lambda_2}[\lambda_3]\Big)$$
$$=[\lambda'_1]x_1+[\lambda'_2]x_2+[\lambda'_3]x_3$$

□

Thus equation 10 is proved. In the same way we show 11.

6 Polynomial basis

Proposition 3 .
The family $\mathcal{B}_q^n=\{B_\lambda^n(\tau;T,q),\ |\lambda|=n\}$ of q-Bernstein polynomials, forms a basis for the space of polynomials of total degree m ($m\le n$) on T, denoted: $\mathbb{P}_n(T)$.

Proof. (We use the elevation of the degree for the demonstration)
In [2] it's clear that
card($\mathcal{B}_q^n$)= $\binom{n+2}{2}$=dim($\mathbb{P}_n(T)$).
Therefore, it remain to show that $\mathcal{B}_q^n$ is a generating family of the space $\mathbb{P}_n(T)$, i.e
($x^\nu y^\mu \in \mathcal{B}_q^n,\ 0\le\nu+\mu\le n$).
we have $\sum_{|\lambda|=n}B_\lambda^n(\tau;T,q)=1$
thus $1\in\mathcal{B}_q^n$. from 10 and 11, x and y are also in the space spanned by B_q^n.
Now suppose,the proposition is true for $n-1$, there are then coefficients $c_\lambda(q)$, $|\lambda|=n-1$, which depend only on q such that
$x^{\nu-1}y^\mu=\sum_{|\lambda|=n-1}c_\lambda(q)B_\lambda^{n-1}(\tau;T,q)$
as $x=\tau_1x_1+\tau_2x_2+\tau_3x_3$
then

$$x^\nu y^\mu=$$
$$(\tau_1x_1+\tau_2x_2+\tau_3x_3)\sum_{|\lambda|=n-1}c_\lambda(q)B_\lambda^{n-1}(\tau;T,q).$$

We deduce that

$$x^\nu y^\mu=\sum_{|\lambda|=n-1}\frac{x_1}{\delta}c_\lambda(q)\delta\tau_1B_\lambda^{n-1}(\tau;T,q)+$$
$$\sum_{|\lambda|=n-1}\frac{x_2}{\delta}c_\lambda(q)\delta\tau_2B_\lambda^{n-1}(\tau;T,q)+$$
$$\sum_{|\lambda|=n-1}\frac{x_3}{\delta}c_\lambda(q)\delta\tau_3B_\lambda^{n-1}(\tau;T,q),$$

giving

$$x^\nu y^\mu=$$
$$\sum_{|\lambda|=n-1}\frac{x_1}{\delta}c_\lambda(q)\Big(\delta\tau_1+\delta\tau_2+\delta\tau_3-$$
$$(\delta\tau_2+\delta\tau_3)q^{\lambda_1}\Big)\times$$
$$B_\lambda^{n-1}(\tau;T,q)-$$
$$\sum_{|\lambda|=n-1}\frac{x_1}{\delta}c_\lambda(q)\Big(\delta\tau_2+\delta\tau_3-(\delta\tau_2+\delta\tau_3)q^{\lambda_1}\Big)\times$$
$$B_\lambda^{n-1}(\tau;T,q)+$$
$$\sum_{|\lambda|=n-1}\frac{x_2}{\delta}c_\lambda(q)\Big(\delta\tau_2+\delta\tau_3-\delta\tau_3q^{\lambda_2}\Big)\times$$
$$B_\lambda^{n-1}(\tau;T,q)-$$
$$\sum_{|\lambda|=n-1}\frac{x_2}{\delta}c_\lambda(q)\Big(\delta\tau_3-\delta\tau_3q^{\lambda_2}\Big)B_\lambda^{n-1}(\tau;T,q)+$$
$$\sum_{|\lambda|=n-1}x_3c_\lambda(q)\tau_3B_\lambda^{n-1}(\tau;T,q)$$
$$=\sum_{|\lambda|=n-1}x_1c_\lambda(q)\Big(1-(1-\tau_1)q^{\lambda_1}\Big)B_\lambda^{n-1}(\tau;T,q)-$$
$$\sum_{|\lambda|=n-1}x_1c_\lambda(q)\Big(1-\tau_1-(1-\tau_1)q^{\lambda_1}\Big)\times$$
$$B_\lambda^{n-1}(\tau;T,q)+$$
$$\sum_{|\lambda|=n-1}x_2c_\lambda(q)\Big(1-\tau_1-\tau_3q^{\lambda_2}\Big)B_\lambda^{n-1}(\tau;T,q)-$$
$$\sum_{|\lambda|=n-1}x_2c_\lambda(q)\Big(\tau_3-\tau_3q^{\lambda_2}\Big)B_\lambda^{n-1}(\tau;T,q)+$$
$$\sum_{|\lambda|=n-1}x_3c_\lambda(q)\tau_3B_\lambda^{n-1}(\tau;T,q),$$

Therefore

$$
\begin{aligned}
x^\nu y^\mu = & \sum_{|\lambda|=n-1} \frac{[\lambda_1+1]}{[n]} x_1 c_\lambda(q) B^n_{\lambda+\epsilon^1}(\tau;T,q) + \\
& \sum_{|\lambda|=n-1} \frac{[\lambda_2+1]}{[n]} x_2 c_\lambda(q) B^n_{\lambda+\epsilon^2}(\tau;T,q) + \\
& \sum_{|\lambda|=n-1} \frac{[\lambda_3+1]}{[n]} x_3 c_\lambda(q) B^n_{\lambda+\epsilon^3}(\tau;T,q) - \\
& \sum_{|\lambda|=n-1} x_1 c_\lambda(q) \Big(1-\tau_1-(1-\tau_1)q^{\lambda_1}\Big) \times \\
& B^{n-1}_\lambda(\tau;T,q) - \\
& \sum_{|\lambda|=n-1} \frac{x_2}{\delta} c_\lambda(q) \Big(\tau_3-\tau_3 q^{\lambda_2}\Big) B^{n-1}_\lambda(\tau;T,q)
\end{aligned}
$$

we deduce

$$
\begin{aligned}
x^\nu y^\mu = & \sum_{|\lambda|=n-1} d^1_{\lambda+\epsilon^1}(q) B^n_{\lambda+\epsilon^1}(\tau;T,q) + \\
& \sum_{|\lambda|=n-1} d^1_{\lambda+\epsilon^2}(q) B^n_{\lambda+\epsilon^2}(\tau;T,q) + \\
& \sum_{|\lambda|=n-1} d^1_{\lambda+\epsilon^3}(q) B^n_{\lambda+\epsilon^3}(\tau;T,q) - \\
& \sum_{|\lambda|=n-1} d^2_\lambda(q) B^{n-1}_\lambda(\tau;T,q)
\end{aligned}
$$

with

$$
d^1_{\lambda+\epsilon^i}(q) = \frac{[\lambda_i+1]}{[n]} x_i c_\lambda(q), \quad i=1,2,3
$$

and

$$
\begin{aligned}
d^2_\lambda(q) = & x_1 c_\lambda(q) \Big(1-\tau_1-(1-\tau_1)q^{\lambda_1}\Big) + \\
& x_2 c_\lambda(q) \Big(\tau_3-\tau_3 q^{\lambda_2}\Big)
\end{aligned}
$$

By exchanging the indices we obtain

$$
\begin{aligned}
x^\nu y^\mu = & \sum_{|\lambda|=n} d^1_\lambda(q) B^n_\lambda(\tau;T,q) - \\
& \sum_{|\lambda|=n-1} d^2_\lambda(q) B^{n-1}_\lambda(\tau;T,q)
\end{aligned}
$$

By degree elevation, there are some coefficients $d^3_\lambda(q)$, $|\lambda|=n$ such that:

$$
\sum_{|\lambda|=n-1} d^2_\lambda(q) B^{n-1}_\lambda(\tau;T,q) = \sum_{|\lambda|=n} d^3_\lambda(q) B^n_\lambda(\tau;T,q)
$$

We deduce:

$$
x^\nu y^\mu = \sum_{|\lambda|=n} d_\lambda(q) B^n_\lambda(\tau;T,q)
$$

with

$$
d_\lambda(q) = d^1_\lambda(q) - d^3_\lambda(q)
$$

□

7 q-Bézier patches over triangles and q-de Casteljau algorithm

Here we define the patch q-Bernstein Bézier (q-BB) of degree n for a triangle and the associated q-de Casteljau algorithm for computing the image of any point on the triangle.

Let H be a parametric surface defined by

$$
H(\tau) = \sum_{|\lambda|=n} P_\lambda B^n_\lambda(\tau;T,q), \tag{12}
$$

where $\tau = (\tau_1,\tau_2,\tau_3)$ are the barycentric coordinates of a point V with respect to the triangle T and $P_\lambda \in \mathbb{R}^3$, $|\lambda| = n$, are the $\binom{n+2}{n}$ control points of the patch q-BB.

Proposition 4 *let $H(\tau) = \sum_{|\lambda|=n} P_\lambda B^n_\lambda(\tau;T,q)$ be a patch q-BB on a triangle T. Then, for any point V of barycentric coordinates τ in T, the value $H(\tau)$ is recursively computed as follows:*

$$
H(\tau) = \sum_{|\lambda|=n-r} P^r_\lambda(\tau) B^{n-r}_\lambda(\tau;T,q) \tag{13}
$$

where $P^0_\lambda(\tau) := P_\lambda$,
and for $r = 1,\cdots,n$; $|\lambda| = n-r$,

$$
\begin{aligned}
P^r_\lambda(\tau) := & \Big(1-(1-\tau_1)q^{\lambda_1+1}\Big) P^{r-1}_{\lambda+\epsilon^1}(\tau) + \\
& q^{\lambda_1}\Big(1-\tau_1-\tau_3 q^{\lambda_2+1}\Big) P^{r-1}_{\lambda+\epsilon^2}(\tau) + \\
& q^{\lambda_1+\lambda_2}\tau_3 P^{r-1}_{\lambda+\epsilon^3}(\tau)
\end{aligned}
$$

then $P^n_{0,0,0} := H(\tau)$

Proof.

$$H(\tau) = \sum_{|\lambda|=n-r} P_\lambda^r(\tau) B_\lambda^{n-r}(\tau; T, q)$$

$$H(\tau) = \sum_{|\lambda'|=n-r+1} P_{\lambda'}^{r-1} B_{\lambda'}^{n-r+1}(\tau; T, q)$$

applying the recurrence formula, we obtain:

$$H(\tau) = \sum_{|\lambda'|=n-r+1} P_{\lambda'}^{r-1} \Bigg[\left(1-(1-\tau_1)q^{\lambda'_1}\right) \times B_{\lambda'-\epsilon^1}^{n-r}(\tau; T, q) + q^{\lambda'_1}\left(1-\tau_1-\tau_3 q^{\lambda'_2}\right) B_{\lambda'-\epsilon^2}^{n-r}(\tau; T, q) + q^{\lambda'_1+\lambda'_2}\tau_3 B_{\lambda'-\epsilon^3}^{n-r}(\tau; T, q) \Bigg]$$

$$= \sum_{|\lambda'|=n-r+1} \Bigg(P_{\lambda'}^{r-1}\left(1-(1-\tau_1)q^{\lambda'_1}\right) \times B_{\lambda'-\epsilon^1}^{n-r}(\tau; T, q) + q^{\lambda'_1} P_{\lambda'}^{r-1}\left(1-\tau_1-\tau_3 q^{\lambda'_2}\right) B_{\lambda'-\epsilon^2}^{n-r}(\tau; T, q) + P_{\lambda'}^{r-1} q^{\lambda'_1+\lambda'_2}\tau_3 B_{\lambda'-\epsilon^3}^{n-r}(\tau; T, q) \Bigg)$$

by a changing of variable, we have:

$$H(\tau) = \sum_{|\lambda|=n-r} \Bigg(P_{\lambda+\epsilon^1}^{r-1}\left(1-(1-\tau_1)q^{\lambda_1+1}\right) \times B_{\lambda}^{n-r}(\tau; T, q) + q^{\lambda_1} P_{\lambda+\epsilon^2}^{r-1}\left(1-\tau_1-\tau_3 q^{\lambda_2+1}\right) B_{\lambda}^{n-r}(\tau; T, q) + P_{\lambda+\epsilon^3}^{r-1} q^{\lambda_1+\lambda_2}\tau_3 B_{\lambda}^{n-r}(\tau; T, q) \Bigg)$$

$$H(\tau) = \sum_{|\lambda|=n-r} \Bigg(P_{\lambda+\epsilon^1}^{r-1}\left(1-(1-\tau_1)q^{\lambda_1+1}\right) + P_{\lambda+\epsilon^2}^{r-1} q^{\lambda_1}\left(1-\tau_1-\tau_3 q^{\lambda_2+1}\right) + P_{\lambda+\epsilon^3}^{r-1} q^{\lambda_1+\lambda_2}\tau_3 \Bigg) \times B_{\lambda}^{n-r}(\tau; T, q)$$

$$H(\tau) = \sum_{|\lambda|=n-r} \Bigg[\left(1-(1-\tau_1)q^{\lambda_1+1}\right) P_{\lambda+\epsilon^1}^{r-1} + q^{\lambda_1}\left(1-\tau_1-\tau_3 q^{\lambda_2+1}\right) P_{\lambda+\epsilon^2}^{r-1} + q^{\lambda_1+\lambda_2}\tau_3 P_{\lambda+\epsilon^3}^{r-1} \Bigg] B_{\lambda}^{n-r}(\tau; T, q)$$

Hence

$$P_\lambda^r = \left(1-(1-\tau_1)q^{\lambda_1+1}\right) P_{\lambda+\epsilon^1}^{r-1} + q^{\lambda_1}\left(1-\tau_1-\tau_3 q^{\lambda_2+1}\right) P_{\lambda+\epsilon^2}^{r-1} + q^{\lambda_1+\lambda_2}\tau_3 P_{\lambda+\epsilon^3}^{r-1}$$

□

8 Conclusion

In this paper, we have defined new q-Bernstein basis functions on a triangulated domain.We then proved some properties verified by these functions, such as the recurrence relation, the partition of unity, the degree raising, we also proved that these function form a basis. Finally, we gave some examples of q-Bernstein Bezier patches. In the future, we will establish Marsden's identity, simulate the finite element family constructed by Haudié et al.[17] and try to generalize this method.

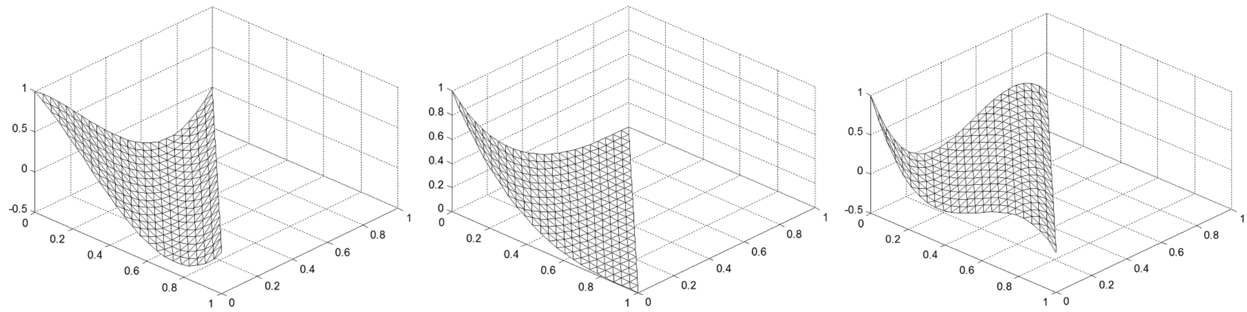

Figure 1: The effect of q-parameter on the cubic q-Bernstein polynomial $B^3_{300}(\tau;T,q)$, from left to right: $q=-1.5$, $q=1$ and $q=2$

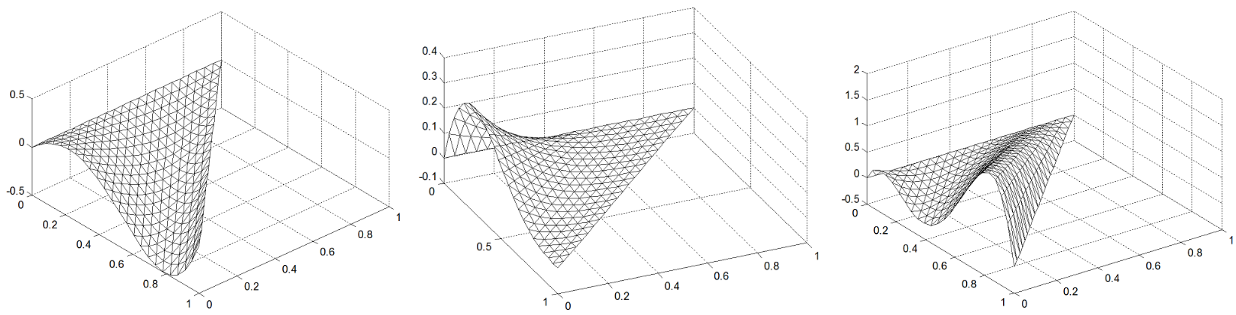

Figure 2: The effect of q-parameter on the cubic q-Bernstein polynomial $B^3_{210}(\tau;T,q)$, from left to right: $q=-1.5$, $q=1$ and $q=2$

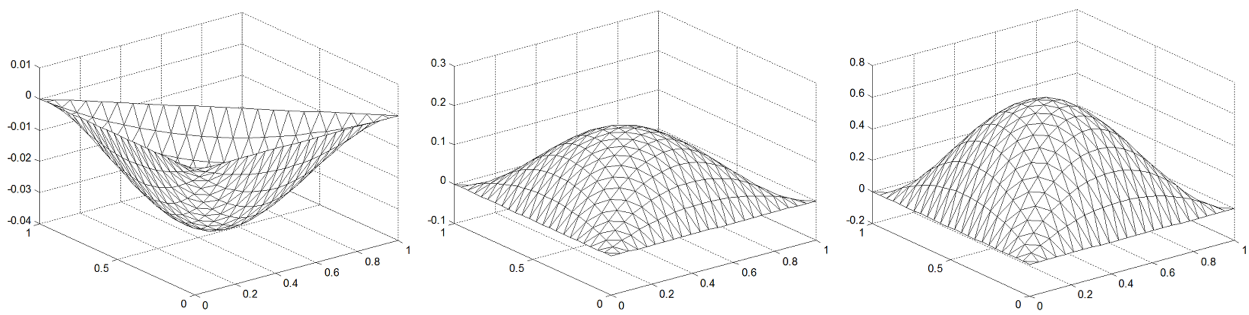

Figure 3: The effect of q-parameter on the cubic q-Bernstein polynomial $B^3_{111}(\tau;T,q)$, from left to right: $q=-1,5$, $q=1$ et $q=2$

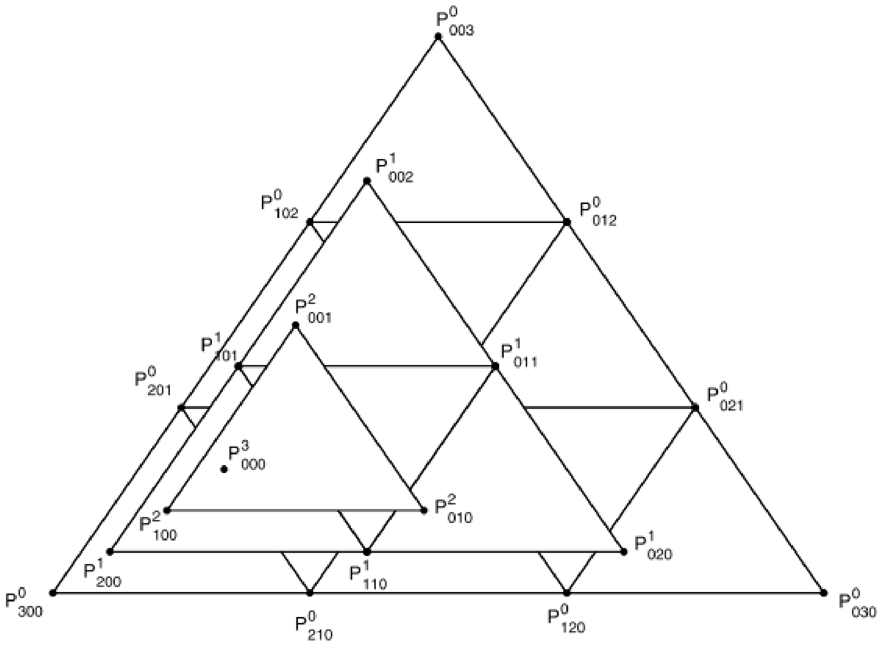

Figure 4: Triangulation describing the q-de Casteljau algorithm for $n = 3$ [14]

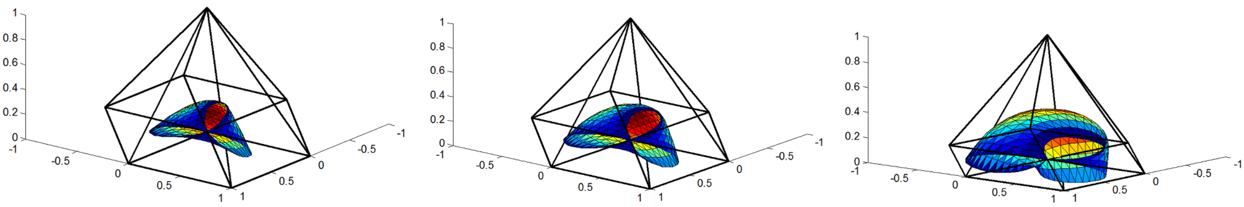

Figure 5: The effect of q-parameter on the cubic q-BB patches. The q values from left to right: $q = 0.7$; $q = 1$ and $q = 1,5$. The controls points are $P_{300} = P_{030} = P_{003} = (0,0,0)$, $P_{210} = (-1,-1,0)$, $P_{120} = (0,-1,0)$, $P_{021} = (1,0,0)$, $P_{012} = (1,1,0)$, $P_{102} = (0,1,0)$, $P_{201} = (-1,0,0)$, $P_{111} = (0,0,1)$.

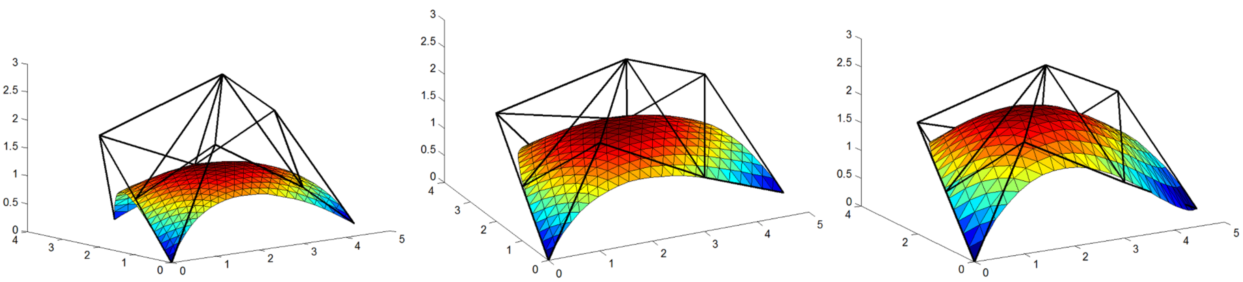

Figure 6: The effect of q-parameter on the cubic q-BB patches. The qq values from left to right: $q = 0.7$; $q = 1$ and $q = 1,5$. The checking points are $P_{300} = (0,0,0)$, $P_{030} = (5,1,0)$, $P_{003} = (2,4,0)$, $P_{210} = (1,0,2)$, $P_{120} = (3,0,1)$, $P_{021} = (4,2,2)$, $P_{012} = (3,3,1)$, $P_{102} = (0,2,2)$, $P_{201} = (0,1,1)$, $P_{111} = (2,1,3)$.

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