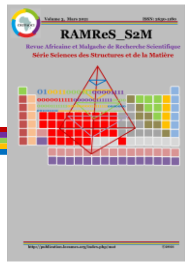
RAMReS Sciences des Structures et de la Matière Vol. 7, N°2 (2023) 11 - 25
http://publication.lecames.org/index.php/mat ISSN: 2630-1180



Existence and uniqueness of entropy solution for nonlinear elliptic equation with nonlinear boundary conditions and measure-data

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Received: 6 April 2023 / Received in revised form: 22 August 2023 / Accepted: 06 September 2023

Abstract:
The aim of this paper is to study, for measure-data, the existence and uniqueness of solutions for nonlinear equation type: $b(u)$ - div $a(u,\ Du) = \nu$, subject to nonlinear boundary conditions of the form $-a(\mathrm{u},\ Du).\eta \in \beta(x,\ u)$. To establish both the existence and uniqueness of the solution, the concept of entropy solution is employed in this study.

Keywords: Non-linear problem ; Entropy solution ; Nonlinear boundary conditions ; Measure-data.
2020 Mathematics Subject Classifications. Primary 35J60, 35J66; Secondary 35R06.

1 Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$ and $1 < p < N$. Consider the nonlinear elliptic problem

$$(E_b)(\nu)\begin{cases} b(\mathrm{u}) - \mathrm{div}\, a(u,\ Du) = \nu & \text{in } \Omega \\ -\langle a(u,\ Du),\ \eta\rangle \in \beta(x,u) & \text{on } \partial\Omega, \end{cases}$$

where η is the unit outward normal vector on $\partial\Omega$, ν is a diffuse measure such that $\nu = \nu\lfloor\Omega$, Du denotes the gradient of u, $b : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, non-decreasing and surjective with $b(0) = 0$ and, for $a.e.x \in \partial\Omega, \beta(x,r) = \partial j(x,r)$ is the subdifferential of a function $j\colon \partial\Omega \times \mathbb{R} \longrightarrow [0,\infty]$ which is convex, lower semicontinuous (l.s.c. for short) in $r \in \mathbb{R}$ for $\sigma - a.e.x \in \partial\Omega$, measurable with respect to the $(\mathrm{N}-1)-$dimensional Hausdorff measure σ on $\partial\Omega$ and such that $j(.,0) = 0$. The vector-valued function $a\colon \mathbb{R}\times\mathbb{R}^N \longrightarrow \mathbb{R}^N$ is continuous satisfying the following classical Leray-Lions-type conditions:

(H_1) – Monotonicity in $\xi \in \mathbb{R}^N$:

$$\big(a(r,\xi) - a(r,\eta)\big).(\xi-\eta) \geq 0\ \forall\, r \in \mathbb{R}, \forall\, \xi,\eta \in \mathbb{R}^{\mathrm{N}}.$$

$(\mathrm{H}_2$ – Coerciveness: $\exists\, \lambda_0 > 0)$ such that

$$\big(a(r,\xi) - a(r,0)\big).\xi \geq \lambda_0|\xi|^p\ \ \forall\, r \in \mathbb{R}, \forall\, \xi \in \mathbb{R}^N.$$

$(\mathrm{H}_3\,)$ – Growth restriction: there exists a continuous function $\Lambda : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such

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that

$$|a(r,\xi)| \leq \Lambda(|r|)(1+|\xi|^{p-1}) \ \forall r \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N.$$

$(H_4)-$ There exists $C : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^+$ continuous such that $\forall r, s \in \mathbb{R}, \forall \xi \in \mathbb{R}^N$,

$$|a(r,\xi) - a(s,\xi)| \leq C(r,s)|r-s|(1+|\xi|^{p-1}).$$

A typical example of a function that satisfies these hypotheses is denoted as $a(r,\xi) = |\xi|^{p-2}\xi + F(r)$, where $F : \mathbb{R} \longrightarrow \mathbb{R}^N$ is a locally Lipschitz function. There have been many findings in the variational setting for Dirichlet or Dirichlet-Neumann problems concerning elliptic problems (refer to [1–7]). In the context of L^1-setting, new concepts of solution such that entropy and renormalized solutions have been introduced for elliptic and parabolic equations in divergence form (see [8–11]). In [4], the authors used and extended the methods introduced in [6] to study the problem

$$\begin{cases} b(u) - \text{div } a(u,Du) = f & \text{in } \Omega \\ -a(u,Du).\eta \in \beta(x,u) & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where a is a divergentiel operator depending of u and β depending also on the space variable x. $b : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, nondecreasing and surjective with $b(0) = 0$, and $f \in L^1(\Omega)$.
In [12], Ouédraogo studied the same kind of problem as in [6], but with measure data instead of L^1-data. Namely, the author proved the existence and the uniqueness of the entropy solution of the problem

$$\begin{cases} u - \text{div } a(u,Du) = \nu & \text{in } \Omega \\ -a(u,Du).\eta \in \beta(x,u) & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where ν is a diffuse measure such that $\nu = \nu\lfloor\Omega$.
In the present paper, we use the same boundary conditions as in [12], but equation $b(u) - \text{div } a(u,Du) = \nu$ is more general as $b : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, nondecreasing and surjective with $b(0) = 0$.
The first difficulty that we encounter in the study of problem $(E_b)(\nu)$ is the fact that the function b is not invertible. So, the a priori estimates on entropy solution are not easy to obtain. To overcome this difficulty, we use the main section of b^{-1} that will be defied later.
The second difficulty is that, when one uses the integration by parts formula in the variational approach (see section 3 below), it appear at the boundary, for the part of the measure data which is in $W^{1,p}(\Omega)$, a term which cannot vanish. In order to treat this difficulty, we use the same arguments as in [12]: we consider a smooth domain Ω in order to work with the space $W_0^{1,p}(\Omega)$ and to going back later to the space $W^{1,p}(\Omega)$. More precisely, Ω is assumed to be a bounded domain in $\mathbb{R}^N$ with a boundary $\partial\Omega$ of class C^1. Then, Ω is an extension domain (see [13]), so we can fix an open bounded subset U_Ω of $\mathbb{R}^N$ such that $\overline{\Omega} \subset U_\Omega$, and there exists a bounded linear operator

$$E : W^{1,p}(\Omega) \to W_0^{1,p}(U_\Omega),$$

for which
i) $E(u) = u$ a.e. in Ω for each $u \in W^{1,p}(\Omega)$,
ii) $\|E(u)\|_{W_0^{1,p}(U_\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$, where C is a constant depending only on Ω.
In this paper, $\mathfrak{M}_B^p(\Omega)$ denote the set of all Radon measures with bounded variation on Ω such that

$$\mathfrak{M}_B^p(\Omega) := \{\nu \in \mathcal{M}_B^p(U_\Omega) :\ \nu = \nu\lfloor\Omega\}.$$

This definition is independent of the open set U_Ω. Note that for $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $\nu \in \mathfrak{M}_B^p(\Omega)$, we have

$$\langle \nu, E(u) \rangle = \int_\Omega u \, d\nu. \tag{3}$$

On the other hand, as ν is diffuse (see [11]), there exists $f \in L^1(U_\Omega)$ and
$F \in (L^{p'}(U_\Omega))^N$ such that $\nu = f-$ div(F) in $\mathcal{D}'(U_\Omega)$. Therefore, we can also write

$$\langle \nu, E(u) \rangle = \int_{U_\Omega} f E(u)\, dx + \int_{U_\Omega} F.\nabla E(u)\, dx. \tag{4}$$

The rest of the paper is organized as follows. In the next section we make precise the notations which will be used in the sequel and recall some facts on measures and capacities. In section 3, we study the problem $(E_b)(\nu)$ by variational methods. We introduce an accretive operator $A_{\delta,b}$ related to problem $(E_b)(\nu)$ and show that $R(I+\alpha A_{\delta,b}) \supset L^\infty(\Omega)$ for all $\alpha > 0$. In section 4, we introduce the notion of entropy solution and prove the existence and uniqueness of this solution. In order to do this, we characterize $\mathcal{A}_b$, the limit of the operator $A_{\delta,b}$ in $\mathcal{M}^p_B(\Omega)$.

2 Function spaces and notation

In this section, we will introduce some notations and definitions that are employed throughout this paper. The N-dimensional Lebesgue measure in $\mathbb{R}^N$ and the $(N-1)$-dimensional Hausdorff measure of $\partial\Omega$ are denoted by $|.|$ and $d\sigma$, respectively. The norm in $L^p(\Omega)$ is represented as $||.||_p$, where $1 \le p \le \infty$. The classical Sobolev space $W^{1,p}(\Omega)$ endowed with the usual norm denoted $||.||_{1,p}$. It is widely recognized (see [14, 15]) that if $u \in W^{1,p}(\Omega)$, the trace of u on $\partial\Omega$ can be defined by the continuous linear trace operator $\tau : W^{1,p}(\Omega) \longrightarrow W^{-\frac{1}{p'},p}(\partial\Omega)$ is surjective.

For $0 < q < \infty$, $\mathcal{M}^q(\Omega)$ is the Marcinkiewicz space (cf. [16]) defined as the set of measurable functions
$g : \Omega \longrightarrow \mathbb{R}$ such that

$$|\{x \in \Omega : |g(x)| > k\}| \le ck^{-q}, c < \infty.$$

As usual, for $k > 0$, we denote by T_k, the truncation function at height $k \ge 0$ defined by

$$T_k(u) = \begin{cases} -k & \text{if } u < -k \\ u & \text{if } |u| \le k \\ k & \text{if } u > k. \end{cases}$$

Let γ be a maximal monotone operator defined on $\mathbb{R}$. We recall the definition of the main section γ_0 of γ:

$$\gamma_0(s) = \begin{cases} \text{the element of minimal absolute} \\ \text{ value of } \gamma(s) \quad \text{if } \gamma(s) \neq \emptyset, \\ +\infty \quad \text{if } [s, +\infty) \cap D(\gamma) = \emptyset, \\ -\infty \quad \text{if } (-\infty, s] \cap D(\gamma) = \emptyset. \end{cases}$$

We denote by $\bar{u}$ the average of u, i.e.
$\bar{u} = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega u(x)\,dx.$
We note $\mathcal{P} = \{S \in C^1(\mathbb{R})/S(0) = 0, 0 \le S' \le 1, \mathrm{supp}(S') \text{ is compact}\}$.
Let $\mathcal{A}$ be a multi-valued operator in $L^1(\Omega)$. Then $\mathcal{A}$ is said to be accretive in $L^1(\Omega)$ if $||u-\tilde{u}||_1 \le ||u-\tilde{u}+\alpha(v-\tilde{v})||_1$ for any $(u,v),(\tilde{u},\tilde{v}) \in \mathcal{A}; \alpha > 0$ i.e., for any $\alpha > 0$, the resolvent of $\mathcal{A}$, $(I+\alpha\mathcal{A})^{-1}$, is a single-valued operator and a contraction in L^1-norm. $\mathcal{A}$ is called $T-$accretive if $||(u-\tilde{u})^+||_1 \le ||(u-\tilde{u}+\alpha(v-\tilde{v}))^+||_1$ for any $(u,v),(\tilde{u},\tilde{v}) \in \mathcal{A}$ and for any $\alpha > 0$. Finally, $\mathcal{A}$ is called $m-$accretive (resp. $m-T-$accretive) in $L^1(\Omega)$ if $\mathcal{A}$ is accretive ($T-$accretive) and moreover, $R(I+\alpha\mathcal{A}) = L^1(\Omega)$ for any $\alpha > 0$ (cf. [17–19] for more details about accretive operators and nonlinear semigroups).

Now, let us introduce some notations and recall some facts about capacities and measures usued throughout this paper (cf. [20–23]). Let G be an arbitrary fixed bounded open subset of $\mathbb{R}^N$ with $\bar{\Omega} \subset G$. Given a compact subset $K \subseteq G$, we define the $p-$capacity of K by:

$$C_{1,p}(K) := \inf\{||\varphi||_{1,p}; \varphi \in C_c^\infty(G), \varphi \ge \chi_K\}.$$

The $p-$capacity of an open set $O \subset G$ is then defined by

$$C_{1,p}(O) := \sup\{C_{1,p}(K); K \subset O, K \text{ is compact}\}$$

which reveals to be equal to the quantity

$$\inf\{||\varphi||_{1,p}; \varphi \in W_0^{1,p}(G), \varphi \ge \chi_O \text{ a.e. on } G\}.$$

Finally, the $p-$capacity of an arbitrary subset $E \subseteq G$ is defined by

$$C_{1,p}(E) := \inf\{C_{1,p}(O); O \text{ open}, \ E \subseteq O\}.$$

Let $\mathcal{M}_B(\Omega)$ (resp. $\mathcal{M}_B(\partial\Omega)$) be the space of all Radon measures on Ω (resp. $\partial\Omega$) with bounded total variation.
For $\mu \in \mathcal{M}_B(\partial\Omega)$, denote by μ^+, μ^- and $|\mu|$ the positive part, negative part and the total variation of the measure μ, respectively, and denote by $\mu = \mu_r d\sigma + \mu_s$ the Radon-Nikodym decomposition of μ relatively to the $(N-1)-$dimensional Hausdorff measure $d\sigma$.
We denote by $\mathcal{M}_B^p(\Omega)$ (resp. $\mathcal{M}_B^p(\partial\Omega)$) the set of Radon measures μ which satisfy $\mu(B) = 0$ for every Borel set $B \subseteq \Omega$ (resp. $B \subseteq \partial\Omega$) such that $C_{1,p}(B) = 0$, i.e. the Radon measures which do not charge sets of 0-capacity.
We denote $\mathcal{J}_0(\partial\Omega) = \{j/j : \partial\Omega \times \mathbb{R} \longrightarrow [0; +\infty]$, such that $j(.,r)$ is $\sigma-$ measurable $\forall r \in \mathbb{R}$ and $j(x,.)$ is convex, l.s.c. satisfying $j(x,0) = 0$ for a.e. $x \in \partial\Omega\}$. For a.e. $x \in \partial\Omega$, we define

$$\begin{aligned} \mathcal{J} : W^{\frac{1}{p'},p}(\partial\Omega) \cap L^\infty(\partial\Omega) &\longrightarrow [0,\infty] \\ u &\longmapsto \int_{\partial\Omega} j(.,u)d\sigma. \end{aligned}$$

Note that $\mathcal{J}$ naturally extends to a functional $\hat{\mathcal{J}}$ on $W_0^{1,p}(G) \cap L^\infty(G)$ as follows: $\hat{\mathcal{J}}(u) = \int_{\partial\Omega} j(.,\tau(u))d\sigma$ for any $u \in W_0^{1,p}(G)$. We recall that the closure of $D(\hat{\mathcal{J}})$ in $W_0^{1,p}(G)$ is a convex bilateral set, so according to [24], there exist unique (in the sense q.e.) functions γ_+, γ_- which are cap-quasi-l.s.c. and cap-quasi-u.s.c. respectively, such that
$\overline{D(\mathcal{J})}^{||.||_{\frac{1}{p'},p}} = \{u \in W^{\frac{1}{p'},p}(\partial\Omega); \gamma_-(x) \leq \tilde{u}(x) \leq \gamma_+(x) \text{ q.e. on } \partial\Omega\}$.
Moreover, $\gamma_-(x) = \inf_n \tilde{u}_n(x) = \lim_n \inf_{1\leq k\leq n} \tilde{u}_k(x)$ q.e. $x \in \partial\Omega$ (respectively the corresponding analogue for γ_+) for any $||.||_{\frac{1}{p'},p}-$dense sequence $(u_n)_n$ in $D(\mathcal{J})$. We define the subdifferential operator:
$\partial\mathcal{J} \subseteq \left(W^{\frac{1}{p'},p}(\partial\Omega) \cap L^\infty(\partial\Omega)\right) \times \left(W^{\frac{-1}{p'},p'}(\partial\Omega) + (L^\infty(\partial\Omega))^*\right)$ by

$$\mu \in \partial\mathcal{J}(u) \Longleftrightarrow \begin{cases} u \in W^{\frac{1}{p'},p}(\partial\Omega) \cap L^\infty(\partial\Omega), \\ \mu \in W^{\frac{-1}{p'},p'}(\partial\Omega) + (L^\infty(\partial\Omega))^* \\ \text{and } \mathcal{J}(w) \geq \mathcal{J}(u) + \langle \mu, w-u \rangle, \\ \qquad \forall w \in W^{\frac{1}{p'},p}(\partial\Omega) \cap L^\infty(\partial\Omega), \end{cases}$$

where, here as in the following, if note explicitly stated otherwise, $\langle .,. \rangle$ denotes the duality between $W^{\frac{1}{p'},p}(\partial\Omega) \cap L^\infty(\partial\Omega)$ and its dual.
To end this section, we define the following spaces similar to that introduced in [9, 25]. We denote

$$\mathcal{T}^{1,p}(\Omega) := \{u : \Omega \longrightarrow \mathbb{R} \text{ measurable };\ T_k(u) \in W^{1,p}(\Omega) \text{ for all } k > 0\Big\}.$$

In [25], the author proved that for $u \in \mathcal{T}^{1,p}(\Omega)$, there exists a unique measurable function $w : \Omega \longrightarrow \mathbb{R}$ such that $DT_k(u) = w\chi_{\{|w|<k\}}$ $\forall k > 0$. This function w will be denoted by Du.

Denote by $\mathcal{T}_{tr}^{1,p}(\Omega)$ the subset of $\mathcal{T}^{1,p}(\Omega)$ consisting of the function that can be approximated by functions of $W^{1,p}(\Omega)$ in the following sense: a function $u \in \mathcal{T}^{1,p}(\Omega)$ belongs to $\mathcal{T}_{tr}^{1,p}(\Omega)$ if there exists a sequence $(u_\delta)_\delta \in W^{1,p}(\Omega)$ such that:
(i) $u_\delta \longrightarrow u$ a.e. in Ω;
(ii) $DT_k(u_\delta) \rightharpoonup DT_k(u)$ weakly in $L^1(\Omega)$ for any $k > 0$;
(iii) there exists a measurable function $v : \partial\Omega \longrightarrow \mathbb{R}$ such that $(\tau(u_\delta))_\delta$ converges a.e. in $\partial\Omega$ to v. The function v is called the trace of u, denoted $\tau(u)$ or u.

3 Penalization problem

In this section, we focus on examining an approximate problem of $(E_b)(\nu)$ by including a penalization term $\delta\Lambda$ for a fixed δ. To establish the existence of a variational solution, we introduce an operator $A_{\delta,b}$ and prove that it is surjective. To get an $L^\infty-$ estimate on the approximate solution, we use the similar arguments as in [6,12]: We first redefine and extend the function Λ which appears in hypothesis (H_3), on an odd monotone function ψ on $\mathbb{R}$ such that $\left|\dfrac{a(k,0)}{\psi(k)}\right| \longrightarrow 0$ as $k \longrightarrow \infty$. This will be possible by setting $\Lambda(r) := \sup\limits_{|z|\leq r}\Big\{\psi(|z|), |z||a(z,0)|\Big\}$ for $r \geq 0$. Secondly, we add a penalization term $\delta\psi(u)$ on the boundary for a fixed δ. This allows us to compensate the terme with $a(u,0)$ by choosing k sufficient large such that $\left|\dfrac{a(k,0)}{\psi(k)}\right| < \delta$.

Notice that in our case, we need the surjectivity of the function b and the main section of b^{-1} to conclude.

Now, we define the operator $A_{\delta,b}$ as follows:

$$(b(u),\nu) \in A_{\delta,b} \text{ if and only if}$$

$$u \in W^{1,p}(\Omega)\cap L^{\infty}(\Omega),\ \nu \in \mathfrak{M}^p_B(\Omega)$$

and there exists a measure $\mu \in \mathcal{M}^p_B(\partial\Omega)$ with a.e. $x \in \partial\Omega$,

$$\mu_r(x) \in \partial j(x,u(x)) + \partial I_{[\gamma_-(x),\gamma_+(x)]}(u(x)),$$

such that for all $\phi \in W^{1,p}(\Omega)\cap L^{\infty}(\Omega)$,

$$\int_\Omega a(u,Du).D(u-\phi)dx + \delta\int_{\partial\Omega}\psi(u)(u-\phi)d\sigma \leq \int_\Omega (u-\phi)\,d\nu - \int_{\partial\Omega}(\tilde{u}-\tilde{\phi})\,d\mu,$$

$$\tilde{u} = \gamma_+\mu_s^+\text{- a.e. on } \partial\Omega,\ \tilde{u} = \gamma_-\mu_s^-\text{- a.e. on } \partial\Omega, \tag{5}$$

where for given interval $[a,b]\subset\mathbb{R}$, $I_{[a,b]}$ denotes the convex l.s.c. functional on $\mathbb{R}$ defined by 0 on $[a,b]$, $+\infty$ otherwise.

Remark 3.1. *As the measure $\mu \in \mathcal{M}^p_B(\partial\Omega)$, $|\mu|$ does not charge sets of $0-$capacity. From $|\mu_s| \leq |\mu|$, it follows that $|\mu_s|$ does not charge sets of $0-$capacity. Consequently, the condition (5) is meaningful.*

The Theorem 3.1 below ensures the existence of a variational solution.

Theorem 3.1. *The operator $A_{\delta,b}$ satisfies the following properties:*
i) $A_{\delta,b}$ is $T-$accretive in $L^1(\Omega)$,
ii) $L^\infty(\Omega) \subset R(I+\alpha A_{\delta,b})$ for any $\alpha > 0$,
iii) $D(A_{\delta,b})$ is dense in $L^1(\Omega)$.

Proof. i) As ν is diffuse, there exists $f \in L^1(U_\Omega)$ and $F \in (L^{p'}(U_\Omega))^N$ such that $\nu = f-\operatorname{div}(F)$ in $\mathcal{D}'(U_\Omega)$. Let u, v such that

$$\begin{cases}\nu_1 = f - \operatorname{div}\,(F_1) \in b(u) + A_{\delta,b}u,\\ \nu_2 = g - \operatorname{div}\,(F_2) \in b(v) + A_{\delta,b}v.\end{cases} \tag{6}$$

By employing similar reasoning as in the work [12], we show that

$$\int_\Omega (b(u)-b(v))^+dx \leq \int_\Omega (f-g)^+dx. \tag{7}$$

ii) Without loss of generality, we can assume that $\alpha = 1$ for the purpose of proving that $L^\infty(\Omega) \subset R(I + A_{\delta,b})$. We take ν in $\mathfrak{M}^p_B(\Omega)\cap L^\infty(\Omega)$. Then, there exists $f \in L^\infty(U_\Omega)$ and $F \in (L^{p'}(U_\Omega))^N$ such that

$$\nu = \tilde{f} - \operatorname{div}\,(\tilde{F}) \text{ in } \mathcal{D}'(\Omega),$$

where $\tilde{f} = \chi_\Omega f$ and $\tilde{F} = \chi_\Omega F$.

For $\lambda \geq 1$, we regularized the problem $(E_b)(\nu)$ by problems of the form

$$\begin{cases} b\Big(T_l(u_\lambda)\Big) + \lambda|T_l(u_\lambda)|^{p-2}T_l(u_\lambda) \\ -\operatorname{div}\, a(T_l(u_\lambda),Du_\lambda) = \tilde{f} - \operatorname{div}(\tilde{F}) \text{ in } \Omega,\\ \\ -a(T_l(u_\lambda),Du_\lambda).\eta = \beta_\lambda(x,T_l(u_\lambda)) \\ \quad +\delta T_l(\psi(u_\lambda)) \text{ on } \partial\Omega,\end{cases}$$

where $k \geq (b^{-1})_0\Big(\|\tilde{f}\|_\infty + 1\Big)$, which satisfies $\left|\dfrac{a(k,0)}{\psi(k)}\right| < \delta$ with $(b^{-1})_0$ the main section of b^{-1}. Here, $l > \max\{k,\psi(k)\}$, and $\beta_\lambda(x,.)$ represents the Yosida approximation of $\beta(x,.)$ defined by

$$\beta_\lambda(x,.) = \lambda\Big(I - (I+\frac{1}{\lambda}\beta(x,.))\Big)^{-1}.$$

Let us consider the operator $A_{\delta,\lambda,b} : W^{1,p}(\Omega) \longrightarrow [W^{1,p}(\Omega)]^*$ defined as follows:

$$\langle A_{\delta,\lambda,b}u_\lambda,\phi\rangle = \lambda\int_\Omega |T_l(u_\lambda)|^{p-2}T_l(u_\lambda)\phi dx$$
$$+\int_\Omega a(T_l(u_\lambda),Du_\lambda).D\phi dx + \int_\Omega b\Big(T_l(u_\lambda)\Big)\phi dx$$
$$+\int_{\partial\Omega}\beta_\lambda(.,T_l(u_\lambda))\phi d\sigma + \delta\int_{\partial\Omega}T_l(\psi(u_\lambda))\phi d\sigma,$$

for all $\phi \in W^{1,p}(\Omega)$, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $W^{1,p}(\Omega)$ and $(W^{1,p}(\Omega))^*$.
This operator is surjective throughout the following result.

Lemma 3.1. *The operator $A_{\delta,\lambda,b}$ is bounded, coercive and verifies the $(M)-$property.*

Proof. The proof of Lemma 3.1 follows the same lines as the proof of Lemma 3.1 in [12]. □

From Lemma 3.1 and according to [26], we deduce that the operator $A_{\delta,\lambda,b}$ is surjective. So, for all $\nu \in \mathfrak{M}^p_B(\Omega)\cap(W^{1,p}(\Omega))^*$ there exists $u_\lambda \in W^{1,p}(\Omega)$ such that for all $\phi \in W^{1,p}(\Omega)$

$$\langle A_{\delta,\lambda,b}\ b(u_\lambda)-\nu, u_\lambda-\phi\rangle \leq 0. \tag{8}$$

Taking $\phi = u_\lambda - p^+_\varepsilon(u_\lambda-k)$ as a test function in (8), where $p^+_\varepsilon(.)$ is an approximation of $sign^+_0(.)$ defined as follow

$$p^+_\varepsilon(r) = \begin{cases} 1 & \text{if } r>\varepsilon \\ \dfrac{1}{\varepsilon}r & \text{if } 0<r<\varepsilon \\ 0 & \text{if } r<0, \end{cases}$$

and using hypothesis (H_2), we obtain

$$\int_\Omega b(T_l(u_\lambda))p^+_\varepsilon(u_\lambda-k)\,dx - \int_\Omega p^+_\varepsilon(u_\lambda-k)\,d\nu$$
$$+\lambda\int_\Omega |u_\lambda|^{p-2}u_\lambda p^+_\varepsilon(u_\lambda-k)\,dx$$
$$+\frac{1}{\varepsilon}\int_{\{k<u_\lambda<k+\varepsilon\}} a(T_l(u_\lambda),0).Du_\lambda\,dx$$
$$\leq -\delta\int_{\partial\Omega}T_l(\psi(u_\lambda))p^+_\varepsilon(u_\lambda-k)\,d\sigma$$
$$-\int_{\partial\Omega}\beta_\lambda(.,T_l(u_\lambda))p^+_\varepsilon(u_\lambda-k)\,d\sigma. \tag{9}$$

Using the same arguments as in [12], we can pass to the limit in (9) as $\varepsilon \longrightarrow 0$ to obtain

$$\int_{\{u_\lambda>k\}} b(T_l(u_\lambda))dx \leq \delta\int_{\partial\Omega\cap\{u_\lambda>k\}} T_l(\psi(u_\lambda))d\sigma$$
$$+\int_{\{u_\lambda>k\}} f_\lambda dx - \delta\int_{\partial\Omega\cap\{u_\lambda>k\}} T_l(\psi(u_\lambda))d\sigma$$
$$\leq \int_{\{u_\lambda>k\}}\tilde{f}dx.$$

Then,

$$\int_{\{u_\lambda>k\}}\Big(b\Big(T_l(u_\lambda)\Big)-b\Big(T_l(k)\Big)\Big)dx$$
$$\leq \int_{\{u_\lambda>k\}}\Big(\tilde{f}-b\Big(T_l(k)\Big)\Big)dx.$$

As $l>k$ then $T_l(k)=k$. Thus, we have $\int_{\{u_\lambda>k\}}\Big(\tilde{f}-b\Big(T_l(k)\Big)\Big) = \int_{\{u_\lambda>k\}}\Big(\tilde{f}-b(k)\Big) \leq 0$ as $k \geq (b^{-1})_0\Big(\|\tilde{f}\|_\infty+1\Big)$. From inequality above, we get $\int_{\{u_\lambda>k\}}\Big[b\Big(T_l(u_\lambda)\Big)-b\Big(T_l(k)\Big)\Big]^+dx \leq 0, \forall l > k$ and then $b(T_l(u_\lambda)) \leq b(k)$ a.e. in $\{u_\lambda > k\}$. We conclude that $b(u_\lambda) \leq b(k)$ a.e. in Ω.
Similarly, we prove that $b(u_\lambda) \geq b(-k)$ a.e. in Ω. Consequently $|b(u_\lambda)| \leq b(k) = C$.
We deduce that $|u_\lambda| \leq C$ (since b is continuous and surjective) and then

$$\|u_\lambda\|_\infty \leq C, \tag{10}$$

where C is a constant depending on ν and b.
Using the same arguments as in [12], we can pass to the limit in (8) with $\lambda \longrightarrow +\infty$, to get

$$\int_\Omega a(u,Du).D\phi\,dx + \delta\int_{\partial\Omega}\psi(u)\phi\,d\sigma$$
$$= \int_\Omega \phi\,d\nu - \int_\Omega b(u)\phi\,dx - \int_{\partial\Omega}\tilde{\phi}\,d\mu, \tag{11}$$

for all $\phi \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ and $\mu \in \partial\mathcal{J}(u)$.
To conclude the proof of *ii*), we prove, using the fact that $\mu \in \partial\mathcal{J}(u)$ and same technics as in [20], (proposition 20) that the measure μ satisfies a.e. $x \in \partial\Omega$,

$$\mu_r(x) \in \partial j(x,u(x)) + \partial I_{[\gamma_-(x),\gamma_+(x)]}(u(x))$$

and

$\tilde{u} = \gamma_- \mu_s^-$-a.e. on $\partial\Omega$, $\tilde{u} = \gamma_+ \mu_s^+$- a.e. on $\partial\Omega$.

iii) We show that $D(A_{\delta,b})$ is dense in $L^1(\Omega)$ i.e. $\overline{D(A_{\delta,b})}^{\|\cdot\|_1} = L^1(\Omega)$.
We have $D(A_{\delta,b}) \subset L^\infty(\Omega) \subset L^1(\Omega)$ (since Ω is bounded). Therefore $\overline{D(A_{\delta,b})}^{\|\cdot\|_1} \subset L^1(\Omega)$.
Reciprocaly, let's show that $L^1(\Omega) \subset \overline{D(A_{\delta,b})}^{\|\cdot\|_1}$. To this end, it suffices to prove that
$L^\infty(\Omega) \subset \overline{D(A_{\delta,b})}^{\|\cdot\|_1}$ (since $L^\infty(\Omega)$ is dense in $L^1(\Omega)$).
Let $\alpha > 0$. Given $\nu \in \mathfrak{M}_B^p(\Omega) \cap L^\infty(\Omega)$, if we set

$$b(u_\alpha) := (I + \alpha A_{\delta,b})^{-1}\nu,$$

then $(b(u_\alpha), \dfrac{1}{\alpha}(\nu - b(u_\alpha))) \in A_{\delta,b}$.
So, taking $\phi = 0$ as a test function in the definition of the operator $A_{\delta,b}$, we get

$$\begin{aligned}\int_\Omega a(u_\alpha, Du_\alpha).Du_\alpha\, dx + \delta \int_{\partial\Omega} \psi(u_\alpha)(u_\alpha)\, d\sigma \\ \leq \frac{1}{\alpha}\int_\Omega u_\alpha\, d\nu - \frac{1}{\alpha}\int_\Omega b(u_\alpha)u_\alpha\, dx \\ - \int_{\partial\Omega} \tilde{u}_\alpha\, d\mu_\alpha. \qquad (12)\end{aligned}$$

Using hypothesis (H_2), we have $\int_\Omega [a(u_\alpha, Du_\alpha) - a(u_\alpha, 0)].Du_\alpha \geq \lambda_0 \parallel Du_\alpha \parallel_p^p$.
Then, we deduce from inequality (12) that

$$\begin{aligned}\lambda_0 ||Du_\alpha||_p^p \leq \frac{1}{\alpha}\int_\Omega u_\alpha\, d\nu - \frac{1}{\alpha}\int_\Omega b(u_\alpha)u_\alpha\, dx \\ -\delta\int_{\partial\Omega} \psi(u_\alpha)u_\alpha\, d\sigma - \int_{\partial\Omega} \tilde{u}_\alpha\, d\mu_\alpha \\ - \int_\Omega a(u_\alpha, 0).Du_\alpha\, dx. \qquad (13)\end{aligned}$$

Using the hypothesis (H_3), the monotonicity of ψ, properties of μ and the $L^\infty-$estimate on u_α, we get from (13)

$$\lambda_0 ||Du_\alpha||_p^p \leq \frac{1}{\alpha}C' + C. \qquad (14)$$

Using the hypothesis (H_3), Hölder inequality and (14), we get

$$\begin{aligned}\alpha \int_\Omega |a(u_\alpha, Du_\alpha)| &\leq \alpha \int_\Omega \Lambda(|u_\alpha|)(1 + |Du_\alpha|^{p-1}) \\ &\leq \alpha C_1 + \alpha\Big(\int_\Omega \big(\Lambda(|u_\alpha|)\big)^p\Big)^{\frac{1}{p}} \Big(\int_\Omega |Du_\alpha|^p\Big)^{\frac{1}{p'}} \\ &\leq \alpha C_1 + \alpha C_2 \Big(\frac{1}{\alpha}C' + C\Big)^{\frac{1}{p'}} \\ &\leq \alpha C_1 + \alpha 2^{\frac{1}{p'}} C_2 \Big(\frac{1}{2}\Big(\frac{C'}{\alpha}\Big)^{\frac{1}{p'}} + \frac{1}{2}C^{\frac{1}{p'}}\Big) \\ &\leq \alpha C_1 + \alpha^{\frac{1}{p}} C_3 + \alpha C_4 \\ &\longrightarrow 0 \text{ as } \alpha \longrightarrow 0.\end{aligned}$$

On the other hand, if $\phi \in D(\Omega)$, taking $u_\alpha + \phi$ and $u_\alpha - \phi$ as test functions in the definition of the operator $A_{\delta,b}$, we get after adding both inequalities

$$\begin{aligned}\alpha \int_\Omega a(u_\alpha, Du_\alpha).D\phi\, dx + \alpha\delta \int_{\partial\Omega} \psi(u_\lambda)\phi\, d\sigma \\ = \int_\Omega \phi\, d\nu - \int_\Omega b(u_\alpha)\phi\, dx - \alpha \int_{\partial\Omega} \tilde{\phi}\, d\mu_\alpha. \quad (15)\end{aligned}$$

Passing to the limit as $\alpha \longrightarrow 0$ in inequality (15), we get

$$\lim_{\alpha \longrightarrow 0} \int_\Omega b(u_\alpha)\phi = \int_\Omega \phi\, d\nu, \quad \forall \phi \in \mathcal{D}(\Omega). \quad (16)$$

Since $(u_\alpha)_\alpha$ is bounded in $L^\infty(\Omega)$, there exists a subsequence $(u_{\alpha_n})_n$ such that $u_{\alpha_n} \rightharpoonup u$ weakly in $L^p(\Omega)$; so $b(u_{\alpha_n}) \rightharpoonup b(u)$. Therefore, using (16), we get $b(u) = \nu$.
As $(u_\alpha)_\alpha$ is bounded in $L^\infty(\Omega)$ and b is continuous, we have

$$||b(u_\alpha)||_p^p = \int_\Omega |b(u_\alpha)|^p \leq \int_\Omega ||b(u_\alpha)||_\infty^p \leq C.$$

By Lebesgue dominated convergence theorem, $b(u_\alpha) \longrightarrow \nu$ in $L^p(\Omega)$. As a consequence, $\nu \in \overline{D(A_{\delta,b})}^{||\cdot||_1}$.
The proof of theorem 3.1 is thus accomplished. □

4 Entropy solution

In accordance with [4], we define an entropy solution of $(E_b)(\nu)$ in the following manner.

Definition 4.1. *A function $u \in \mathcal{T}_{tr}^{1,p}(\Omega)$ is an entropy solution for problem $(E_b)(\nu)$ if $b(u) \in L^1(\Omega)$ and there exists a measure $\mu \in \mathcal{M}_B^p(\partial\Omega)$ with a.e. $x \in \partial\Omega$,*

$$\mu_r(x) \in \partial j(x, u(x)) + \partial I_{[\gamma_-(x),\gamma_+(x)]}(u(x)), \tag{17}$$

such that for all $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega a(u, Du).DT_k(u-\phi)\,dx \le \int_\Omega T_k(u-\phi)\,d\nu - \int_\Omega b(u)T_k(u-\phi)\,dx - \int_{\partial\Omega} T_k(\tilde{u}-\tilde{\phi})\,d\mu,$$

$$\tilde{u} = \gamma_+\mu_s^+\text{- a.e. } \partial\Omega,\ \tilde{u} = \gamma_-\mu_s^-\text{- a.e. } \partial\Omega. \tag{18}$$

We define an operator $\mathcal{A}$ by the rule:

$(b(u), \nu - b(u)) \in \mathcal{A}$ if and only if

$$\begin{cases} \nu \in \mathfrak{M}_B^p(\Omega) \text{ and} \\ u \text{ is an entropy solution of problem } (E_b)(\nu). \end{cases}$$

In the following, we use the notation $A_{m,n}$ (resp. $\psi_{m,n}$) instead of A_δ (resp. $\delta\psi$), where $\psi_{m,n}(u) = \frac{1}{m}\psi(u^+) - \frac{1}{n}\psi(u^-)$, $m, n \in \mathbb{N}^*$.

Theorem 4.1. *The operator $\mathcal{A}$ is $m-$accretive with dense domaine in $L^1(\Omega)$ and*

$$\mathcal{A} = \liminf_{m,n\longrightarrow+\infty} A_{m,n}$$

where $\liminf_{m,n\longrightarrow+\infty} A_{m,n}$ is the operator defined by: $(x,y) \in \liminf_{m,n\longrightarrow+\infty} A_{m,n} \Longleftrightarrow$

$$\begin{cases} \forall m,n > 0,\ (x_{m,n}, y_{m,n}) \in A_{m,n} \text{ and} \\ (x,y) = \liminf_{m,n\longrightarrow+\infty} (x_{m,n}, y_{m,n}) \text{ in } X \times X. \end{cases}$$

Proof. The proof of this theorem is carried out in several steps.

Step 1: A priori estimates.

Let $\nu \in \mathfrak{M}_B^p(\Omega)$. Since ν is diffuse, recall that $\nu = f -$ div (F) in $\mathcal{D}'(U_\Omega)$ with $f \in L^1(U_\Omega)$ and $F \in \left(L^{p'}(U_\Omega)\right)^N$ where U_Ω is the open bounded subset of $\mathbb{R}^N$ which extend Ω via the operator E.

We approximate f and b respectively by $f_{m,n} = (f \wedge m) \vee (-n) \in L^\infty(\Omega)$ nondecreasing in m, nonincreasing in n, and $b_{m,n}(\sigma) = b(\sigma) + \frac{1}{m}\sigma^+ - \frac{1}{n}\sigma^-\ \forall\ \sigma \in \mathbb{R}$.

Note that $||f_{m,n}||_1 \le ||f||_1$.

Let $\left(F_{m,n}\right)_{m,n\ge1} \subset \mathcal{C}_0^\infty(U_\Omega)$ be a sequence such that $F_{m,n} \to F$ strongly in $\left(L^{p'}(U_\Omega)\right)^N$, as $m,n \to +\infty$. For any $m,n \ge 1$ we set

$$\tilde{F}_{m,n} = \chi_\Omega F_{m,n} \text{ and } \nu_{m,n} = f_{m,n} - \mathrm{div}(\tilde{F}_{m,n}).$$

For any $m,n \ge 1$, one has $\nu_{m,n} \in \mathfrak{M}_B^p(\Omega)$, $\nu_{m,n} \rightharpoonup \nu$ in $\mathcal{M}_b(U_\Omega)$ and $\nu_{m,n} \in L^\infty(\Omega)$. Furthermore, for any $k > 0$ and any $\xi \in \mathcal{T}^{1,p}(\Omega)$,

$$\left|\int_\Omega T_k(\xi)\,d\nu_{m,n}\right| \le kC(\nu,\Omega).$$

By Theorem 3.1, $\nu_{m,n} \in R(I + A_{m,n})$ and there exists $u_{m,n} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and a measure $\mu_{m,n} \in \mathcal{M}_B^p(\partial\Omega)$ satisfying

$$(\mu_{m,n})_r(x) \in \partial j(x, u_{m,n}(x)) + +\partial I_{[\gamma_-(x),\gamma_+(x)]}(u_{m,n}(x)),\ \text{a.e. } x \in \partial\Omega,$$

such that for all $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$,

$$\begin{aligned} &\int_\Omega a(u_{m,n}, Du_{m,n}).D(u_{m,n}-\phi)\,dx + \int_{\partial\Omega} \psi_{m,n}(u_{m,n})(u_{m,n}-\phi)\,d\sigma \\ &\le \int_\Omega (u_{m,n}-\phi)\,d\nu_{m,n} - \int_{\partial\Omega}(\tilde{u}_{m,n}-\tilde{\phi})\,d\mu_{m,n} \\ &\quad - \int_\Omega b_{m,n}(u_{m,n})(u_{m,n}-\phi)\,dx \end{aligned} \tag{19}$$

and $\tilde{u}_{m,n}^{+/-} = \gamma_{+/-}\ (\mu_{m,n})_s^{+/-}$ a.e. on $\partial\Omega$.

In the following, let $k > 0$ be fixed. Using $\phi = u_{m,n} - T_k(u_{m,n})$ as a test function in (19) and applying hypothesis (H_2), we obtain

$$\begin{aligned} &\lambda_0 \int_\Omega |DT_k(u_{m,n})|^p + \frac{1}{m}\int_{\partial\Omega} T_k(u_{m,n})\psi(u_{m,n}^+)\,d\sigma \\ &-\frac{1}{n}\int_{\partial\Omega} T_k(u_{m,n})\psi(u_{m,n}^-)\,d\sigma \le \int_\Omega T_k(u_{m,n})\,d\nu_{m,n} \\ &- \int_\Omega T_k(u_{m,n})b_{m,n}(u_{m,n})\,dx - \int_{\partial\Omega} T_k(\tilde{u}_{m,n})d\mu_{m,n} \\ &- \int_\Omega a(u_{m,n},0).DT_k(u_{m,n})\,dx. \end{aligned} \tag{20}$$

By Gauss-Green Formula and hypothesis (H_3), we have

$$\begin{aligned}\Big|\int_\Omega a(u_{m,n},0).DT_k(u_{m,n})\Big| &\le \Big|\int_{\partial\Omega}\Big(\int_0^{T_k(u_{m,n})} a(r,0)dr\Big).\eta d\sigma\Big| \\ &\le \int_{\partial\Omega}\Big|\int_0^{T_k(u_{m,n})}\Lambda(|r|)dr\Big|d\sigma \\ &\le C, \end{aligned} \tag{21}$$

where C is a constant depending on k. Then, from inequality (20), according to the monotonicity of ψ, we conclude

$$\lambda_0\int_\Omega |DT_k(u_{m,n})|^p \ \le\ C. \tag{22}$$

Thus $(T_k(u_{m,n}))_{m,n}$ is a bounded subset of $W^{1,p}(\Omega)$. Hence, after passing to a suitable subsequence if necessary, $(T_k(u_{m,n}))_{m,n}$ is weakly convergent in $W^{1,p}(\Omega)$. Then, $T_k(u_{m,n}) \longrightarrow v_k$ in $L^p(\Omega)$ as $m,n\longrightarrow\infty$. we may also assume that $DT_k(u_{m,n}) \rightharpoonup g_k$ in $(L^p(\Omega))^N$ as $m,n\longrightarrow\infty$.
Now, we must prove the convergence almost everywhere of $u_{m,n}$. As $A_{m,n}$ is $T-$accretive in $L^1(\Omega)$, we have for all $m\ge m'$,

$$\int_\Omega\Big(b_{m',n}(u_{m',n})-b_{m,n}(u_{m,n})\Big)^+dx \le \int_\Omega (f_{m',n}-f_{m,n})^+dx.$$

As $f_{m,n}$ is nondecreasing in m, we have

$$\begin{aligned} m\ge m' &\implies f_{m',n}-f_{m,n}\le 0\\ &\implies (f_{m',n}-f_{m,n})^+=0.\end{aligned}$$

Then
$m\ge m' \Longrightarrow \Big(b_{m',n}(u_{m',n})-b_{m,n}(u_{m,n})\Big)^+ = 0$, i.e. $b_{m',n}(u_{m',n})-b_{m,n}(u_{m,n})\le 0$ a.e. on Ω. Thus,

$$\Big(b(u_{m',n})-b(u_{m,n})\Big)+\frac{1}{m'}\Big((u_{m',n})^+-(u_{m,n})^+\Big) +\frac{1}{n}\Big((u_{m,n})^- -(u_{m',n})^-\Big)\le 0. \tag{23}$$

It is easy to see that the three terms of the inequality (23) have the same sign, then they are negatives which implies that $u_{m',n}-u_{m,n}\le 0$ for $m\ge m'$ and n fixed. Then $(u_{m,n})_m$ is nondecreasing. By the same method, we show that $(u_{m,n})_n$ is nonincreasing.
Since $(u_{m,n})_m$ is uniformly bounded then we deduce that

$$u_{m,n}\uparrow u_n \text{ as } m\to+\infty,\ u_n\downarrow u \text{ as } n\to+\infty.$$

By applying Lebesgue dominated convergence theorem, we get

$$u_{m,n}\uparrow_m u^n\downarrow_n u,\ u_{m,n}\downarrow_n u_m\uparrow_m u \text{ in } L^1(\Omega). \tag{24}$$

Therfore, from (24) we get the convergence of $(u_{m,n})$ to u in $L^1(\Omega)$ and also the convergence almost everywhere on Ω.
Then, we conclude that $v_k = T_k(u)$ and $g_k = DT_k(u)$. Therefore, $T_k(u)\in W^{1,p}(\Omega)$ for all $k>0$. Consequently, $u\in\mathcal{T}^{1,p}(\Omega)$.
Finally, we show exactly as in [9], that $(\tau(u_{m,n}))_{m,n}$ converge a.e. on $\partial\Omega$, and then, $u\in\mathcal{T}^{1,p}_{tr}(\Omega)$.
Step 2: Existence of the measure μ.
We still need to show the existence of a measure $\mu\in\mathcal{M}^p_B(\Omega)$ such that $\mu_{m,n}$ strongly converges to μ in $\mathcal{M}^p_B(\Omega)$.
Consider $u^\lambda_{m,n}$ as a solution to the following equation:

$$\begin{aligned}\int_\Omega a(u^\lambda_{m,n},Du^\lambda_{m,n}).D\varphi\,dx+\frac{1}{m}\int_{\partial\Omega}\psi(u^{\lambda,+}_{m,n})\varphi\,d\sigma -\frac{1}{n}\int_{\partial\Omega}\psi(u^{\lambda,-}_{m,n})\varphi\,d\sigma = \int_\Omega\varphi\,d\nu_{m,n}\\ -\int_\Omega b_{m,n}(u^\lambda_{m,n})\varphi\,dx-\int_{\partial\Omega}\beta_\lambda(.,u^\lambda_{m,n})\varphi,\end{aligned} \tag{25}$$

for all $\phi\in W^{1,p}(\Omega)\cap L^\infty(\Omega)$.
We know from Theorem 3.1 that $\|\beta_\lambda(.,u^\lambda_{m,n})\|_1$ is uniformly bounded by a constant C independent of λ, thus $\beta_\lambda(.,u^\lambda_{m,n})\rightharpoonup\mu_{m,n}$ in $\mathcal{M}^p_B(\partial\Omega)$ as $\lambda\to 0$. Therefore

$$\begin{aligned}\|\mu_{m,n}\|_{\mathcal{M}^p_B(\partial\Omega)} &\le \liminf_{\lambda\to0}\|\beta_\lambda(.,u^\lambda_{m,n})\|_{\mathcal{M}^p_B(\partial\Omega)}\\ &\le C\end{aligned}$$

and we deduce, after extracting a subsequence if necessary that $\mu_{m,n}\rightharpoonup\mu$ weakly in $\mathcal{M}_B^p(\partial\Omega)$ as $m,n\to\infty$. To establish the strong convergence of $\mu_{m,n}$, we use the following comparison result.

Lemma 4.1. *Let $\tilde{m}>m>0$, $\tilde{n}>n>0$, then*

$$u_{m,\tilde{n}}^\lambda \le u_{m,n}^\lambda \le u_{\tilde{m},n}^\lambda \quad a.e.\ in\ \Omega, \tag{26}$$

and a.e. in $\partial\Omega$,

$$\beta_\lambda(.,u_{m,\tilde{n}}^\lambda) \le \beta_\lambda(.,u_{m,n}^\lambda) \le \beta_\lambda(.,u_{\tilde{m},n}^\lambda). \tag{27}$$

Proof. To prove Lemma 4.1, we consider the test functions $\varphi=p_\varepsilon^+(u_{m,n}^\lambda-u_{\tilde{m},n}^\lambda)$ and $\varphi=p_\varepsilon^+(u_{m,\tilde{n}}^\lambda-u_{m,n}^\lambda)$ in equations (25), corresponding to the solutions $u_{m,n}^\lambda$ and $u_{\tilde{m},n}^\lambda$, respectively. By adding both equations and dropping some non-negative terms, we obtain (26) taking the limit as $\varepsilon\to 0$. We use the definition of β_λ to deduce (27). □

Note that for the positive and negative parts, the result of the lemma 4.1 remains true, that is

$$\pm\beta_\lambda(.,u_{m,\tilde{n}}^\lambda)^\pm \le \pm\beta_\lambda(.,u_{m,n}^\lambda)^\pm \le \pm\beta_\lambda(.,u_{\tilde{m},n}^\lambda)^\pm$$

Thus, through the previous convergence result, we have

$$\pm\mu_{m,\tilde{n}}^\pm \le \pm\mu_{m,n}^\pm \le \pm\mu_{\tilde{m},n}^\pm,$$

which is similar to say that the regular and singular parts verify this comparison result. From this we deduce that

$$\mu_{m,n}^+ \uparrow_m \mu_n^+ \text{ in } \mathcal{M}_B^p(\partial\Omega) \text{ as } m\to\infty.$$

Observe that we get the same results for the negative parts. This is the end of the proof of step 2.

Step 3: Convergence results.

We recall that $u_{m,n}$ satisfies, for all $\varphi\in W^{1,p}(\Omega)\cap L^\infty(\Omega)$

$$\begin{aligned}&\int_\Omega a(u_{m,n},Du_{m,n}).D\varphi\,dx+\frac{1}{m}\int_{\partial\Omega}\psi(u_{m,n}^+)\varphi\,d\sigma\\&-\frac{1}{n}\int_{\partial\Omega}\psi(u_{m,n}^-)\varphi\,d\sigma \qquad =\int_\Omega\varphi\,d\nu_{m,n}\\&-\int_\Omega b_{m,n}(u_{m,n})\varphi\,dx-\int_{\partial\Omega}\beta_\lambda(.,u_{m,n})\varphi.\end{aligned} \tag{28}$$

Taking $\varphi=S(u_{m,n}-\phi)$ as a test function in (28), where $S\in\mathcal{P}=\{p\in C^1(\mathbb{R});p(0)=0,\quad 0\le p'\le 1,\quad \operatorname{supp}(p') \text{ is compact}\}$, $\phi\in W^{1,p}(\Omega)\cap L^\infty(\Omega)$, define

$$l=||\phi||_\infty+\max\{|z|,z\in\operatorname{supp}(S')\}$$

and using the same arguments as in [12], we pass to the limit as $m,n\longrightarrow+\infty$ in (28) to get

$$\begin{aligned}&\int_\Omega a(u,Du).DS(u-\phi)\,dx+\int_{\partial\Omega}S(\tilde{u}-\tilde{\phi})d\mu\\&\le\int_\Omega S(u-\phi)\,d\nu-\int_\Omega b(u)S(u-\phi)\,dx,\end{aligned}$$

for all $\phi\in W^{1,p}(\Omega)\cap L^\infty(\Omega)$, and

$$\mu_r\in\partial j(.,u)+\partial I_{[\gamma_-,\gamma_+]}(u) \text{ a.e. in } \partial\Omega,$$
$$\tilde{u}=\gamma_+\ \mu_s^+ \text{ a.e. on } \partial\Omega,\ \tilde{u}=\gamma_-\ \mu_s^- \text{ a.e. on } \partial\Omega.$$

Taking S as an approximation of T_k, we get the desired entropy inequality. Therefore, we have shown that, for all $\nu\in\mathfrak{M}_B^p(\Omega)\cap L^\infty(\Omega)$, $(I+A_{m,n})^{-1}\nu$ converge in $L^1(\Omega)$ to an entropy solution of the problem $(E_b)(\nu)$, hence $\liminf\limits_{m,n\longrightarrow\infty}A_{m,n}\subset\mathcal{A}$. For the inverse inclusion, we refer to the step below.

Step 4: The accretivity of $\mathcal{A}$.

To prove the accretivity of $\mathcal{A}$, we show as in ([4], Theorem 4.1) and as in Theorem 3.1 of section 3,

$$\int_\Omega|b(w)-b(v)|\le\int_\Omega|f-g|, \tag{29}$$

where $f,g\in L^1(\Omega)$ provide from the decomposition of the measures $\nu_1=f-\operatorname{div}(F_1)\in b(w)+\mathcal{A}w$ and $\nu_2=g-\operatorname{div}(F_1)\in b(v)+\mathcal{A}(v)$.

Step 5: $D(\mathcal{A})$ is dense in $L^1(\Omega)$

For this, we show that $L^\infty(\Omega)\subset\overline{D(\mathcal{A})}^{||.||_1}$.

Let $u\in L^\infty(\Omega)$. Consider $u_{m,n}^\alpha$ and u_α, $\alpha>0$ such that

$$\begin{cases}b_{m,n}(u_{m,n}^\alpha)+\alpha A_{m,n}u_{m,n}^\alpha\ni b(u),\\ b(u_\alpha)+\alpha\mathcal{A}u_\alpha\ni b(u).\end{cases} \tag{30}$$

We know from Theorem 3.1 that $D(A_{m,n})$ is dense in $L^1(\Omega)$, then, for all $m,n\in\mathbb{N}^*$, we have

$$b(u_{m,n}^\alpha)\longrightarrow b(u) \text{ in } L^1(\Omega) \text{ as } \alpha\longrightarrow 0.$$

As in ([4], Theorem 4.1), we show that

$$b(u^\alpha_{m,n}) \longrightarrow b(u_\alpha) \text{ in } L^1(\Omega) \text{ as } m,n \longrightarrow \infty.$$

Then, we deduce that $b(u) \in \overline{D(\mathcal{A})}^{\|\cdot\|_1}$. □

Corollary 1. *Under the assumptions of Theorem 4.1, we have the existence and uniqueness of entropy solution $b(u)$ for the problem $(E_b)(\nu)$.*

5 A numerical example

We finally present some numerical results that we obtained by implementing the influence of a parameter. As an example of application, we made numerical experiments with the following data which are realistic in the study of oil and water flow in homogeneous porous media. We work on the domain $\Omega = (0,1) \times (0,1)$. Given that $\beta(x,u) = 0$, $b(u) = u$, and the field $\mathbf{a}$ is expressed as $\mathbf{a}(u, \nabla u) = A(x)\nabla u$, where $A(x)$ is a continuous function satisfying $A(x) \geq M > 0$, we can also set $F = 0$ and $f = \chi_\Omega$ with

$$f(x) = \begin{cases} 1, & \text{if } x \in [0,1] \\ 0, & \text{if } x \notin [0,1] \end{cases}$$

is the Lebesgue measure restricted to $[0,1]$ and therefore, it is absolutely continuous with respect to the Lebesgue measure. We can verify that the hypotheses of Section 1 are satisfied. In this scenario, the problem $(E_b)(\nu)$ formulated on Ω can be stated as follows:

$$\begin{cases} u - div(A(x)\, Du) = 1 \quad in\ \Omega \\ -A(x)\, Du.\eta = 0 \qquad on\ \partial\Omega. \end{cases} \tag{31}$$

We seek to illustrate the influence of parameter $A(x)$ on the numerical simulations of the problem.

Experiment 1: Numerical Illustration of the solution of (31) for $A(x) = \sin(\frac{2\pi}{L}x)$ and changing L.

From examining these graphs, we can conclude that the variation in the value of L has a significant impact on the shape and amplitude of the solution, highlighting the importance of considering the value of L in the analysis of the given problem.

Experiment 2: Here we take $b(u) = u^2$, then keep $\beta(x,u) = 0$, $\mathbf{a}(u, \nabla u) = A(x)\nabla u$, with $A(x)$ a continuous function satisfying $A(x) \geq M > 0$, $F = 0$ and $f = \chi_\Omega$. The problem $(E_b)(\nu)$ which is formulated on Ω can be written as follows:

$$\begin{cases} u^2 - div(A(x)\, Du) = 1 \quad in\ \Omega \\ -A(x)\, Du.\eta = 0 \qquad on\ \partial\Omega, \end{cases} \tag{32}$$

The graphs illustrate the impact of the various $A(x)$ functions on the solutions. We can observe that for $A(x) = \sin(\frac{2\pi}{L}x)$, the solutions exhibit rapid oscillations with low amplitude, reflecting the fast variation of $A(x)$. In contrast, for $A(x) = 10$, the oscillations are less frequent with a slightly higher amplitude due to the constant nature of $A(x)$. Furthermore, for $A(x) = \delta(x)$, the oscillations are even less frequent and have a higher amplitude, indicating a slower variation caused by the constant function $A(x)$ with a higher value.

These graphs illustrate the importance of taking coefficient function characteristics into account when considering problems, as they have a potent influence on the shape and dynamics of the solutions obtained.

6 Concluding remark

The numerical solutions of the equation (31) depend strongly on the value of $A(x)$. The graphs make it possible to visualize this influence and to draw conclusions about the behavior of the system. The first two graphs are obtained by taking two values of $A(x)$ to represent situations where the thermal conductivity of the medium varies along the x axis. Furthermore, when $A(x) = \delta(x)$ and $A(x) = \sin(\frac{2\pi}{L}x)$ are considered, we obtain the corresponding two graphs. In the case where $A(x) = \delta(x)$, the thermal conductivity varies locally according to the function $\delta(x)$. As shown in the graph, the solution follows the same trend as the function $\delta(x)$, with higher values in areas where $\delta(x)$ is higher. This solution can be used to examine the impact of local variations in thermal conductivity on the overall solution. For $A(x) = \sin(\frac{2\pi}{L}x)$, where L represents the length of the variation and can be chosen according to the geometry of the domain, the thermal conductivity varies periodically according to a sinusoidal function. The graph shows that the solution also fits a sinusoidal function with higher amplitude in the regions where the thermal conductivity is higher. This solution can study the effect of periodic variations in the thermal conductivity on the solution.

To sum up, based on the graphs, it is evident that the thermal conductivity of the environment plays a crucial role in solving equation (31). Any local or periodic changes in the thermal conductivity can cause significant variations in the solution. These findings can enhance our comprehension of the underlying physical phenomena and aid in devising more effective thermal control strategies.

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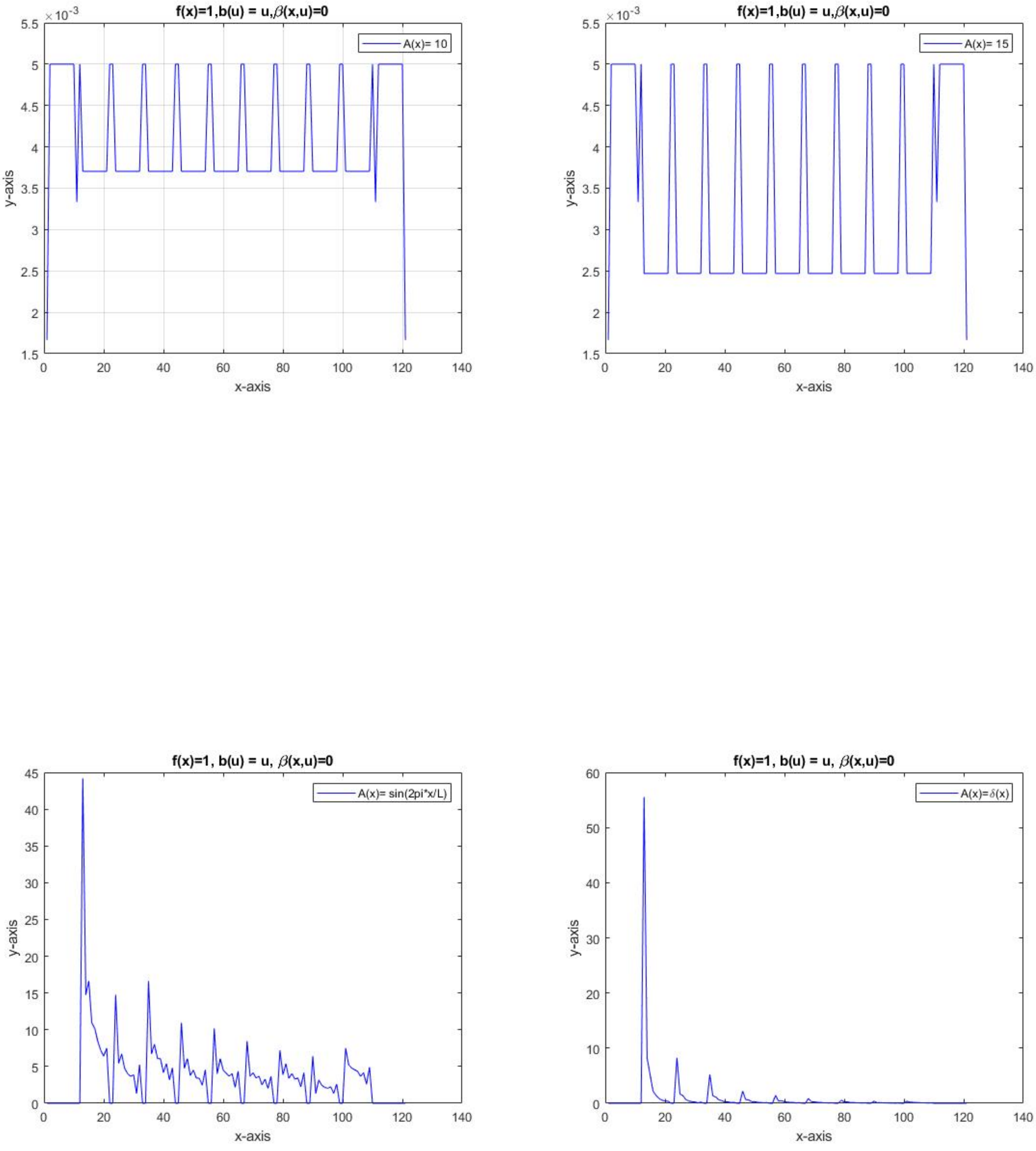

Fig. 1. Impact of the various functions $A(x)$ on the solutions.

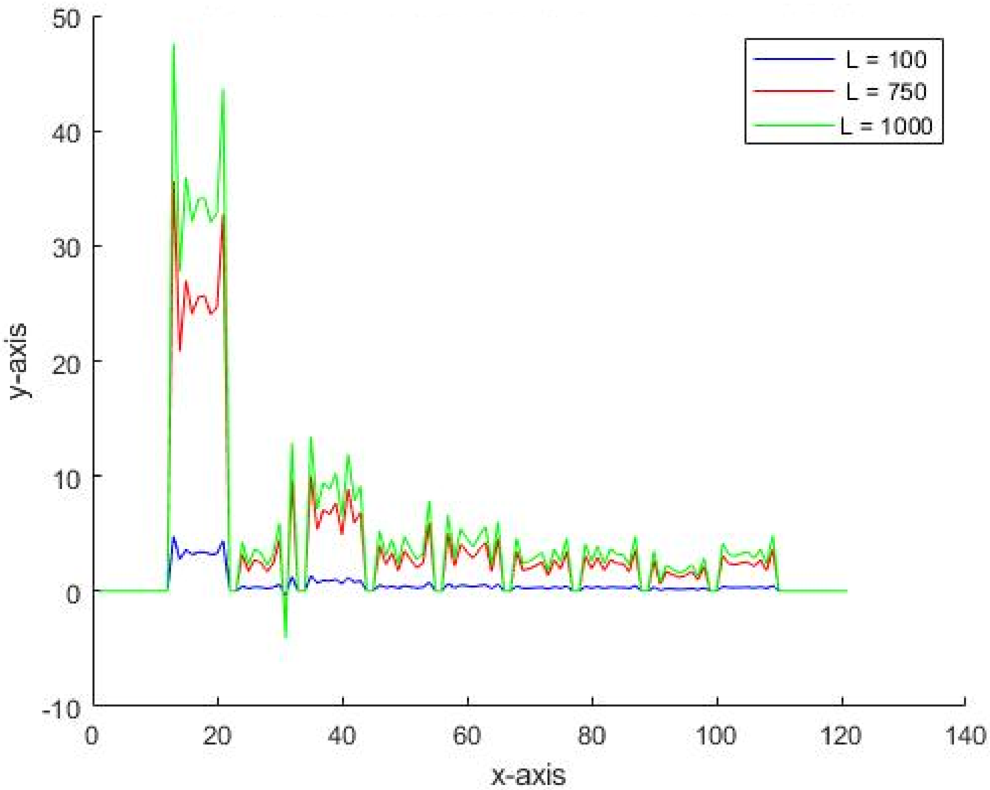

Fig. 2. Comparison of solutions for different L values.

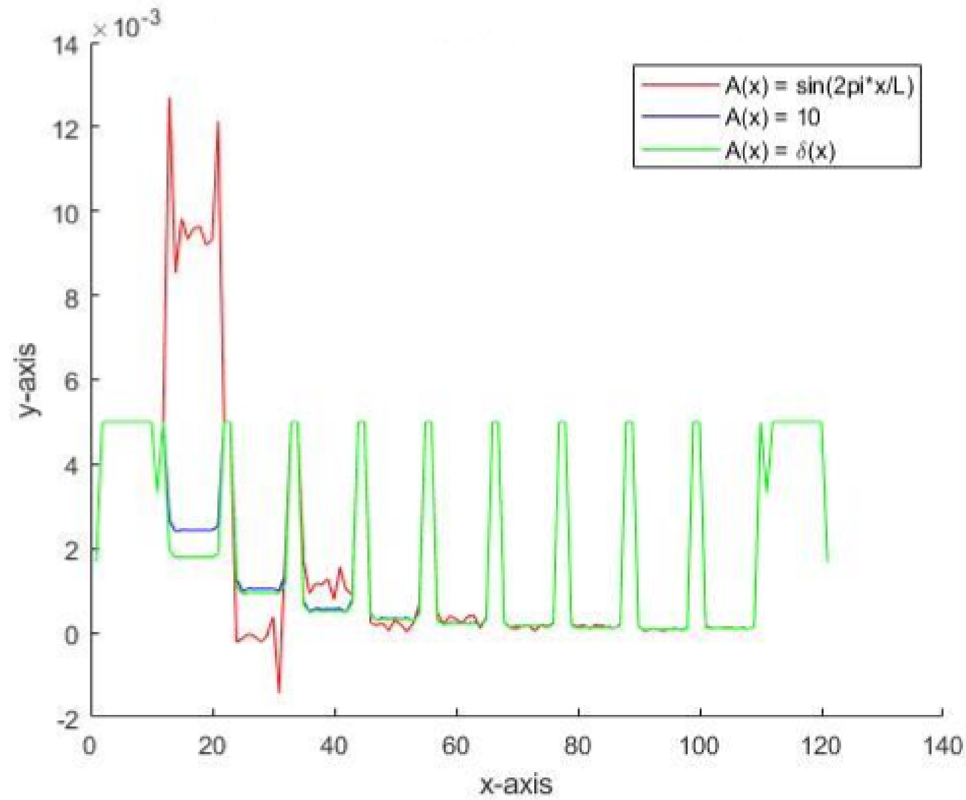

Fig. 3. Comparison of solutions for different functions $A(x)$.