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Analysis of the deterministic and stochastic epidemic models of filariasis

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***Abstract*:**
This article proposes two epidemic models : deterministic and stochastic models of Lymphatic filariasis. For the deterministic model, the basic reproduction number is calculated, the disease-free equilibrium of the model is determined and the stability analysed. If the basic reproduction number is less than one, by using the theorem of Varga (1962) and the standard comparison theoremof Laksmikantham et al. (1989), we have shown that the disease-free equilibrium is globally asymp- totically stable; which means that the disease is eliminated. In the stochastic model, a unique global positive solution for the epidemic model is obtained. We have calculated the threshold parameters which govern the extinction or persistence of the disease. The extinction of the epidemic disease is analysed under assumptions. The persistence in the mean of the stochastic model is also established by building appropriate Lyapunov functions. A comparison of the two models is made. Numerical simulations are carried out to confirm the analytical results.

***Keywords*:** Lymphatic Filariasis; Itô's formula; Extinction; Persistence in the mean.

1 Introduction

Lymphatic filariasis, commonly known as elephantiasis, is a painful and profoundly disfiguring disease. The disease is caused by three species of thread-like nematode worms, known as filariae–Wuchereria bancrofti, Brugia malayi and Brugia timori. Filarial infection can cause a variety of clinical manifestations, including lymphoedema of the limbs, genital disease (hydrocele, chylocele, and swelling of the scrotum and penis) and recurrent acute attacks, which are extremely painful and are accompanied by fever. In communities where filariasis is transmitted, all ages are affected. While the infection may be acquired during childhood its visible manifestations may occur later in life, causing temporary or permanent disability. In endemic countries, lymphatic filariasis has a major social and economic impact with an estimated annual loss

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of one billion and impairing economic activity up to 88% according to the World Health Organization (WHO) 2023. In order to eradicate filariasis, some authors had already conducted research on the subject see [1, 2, 6, 19, 20, 21]. Among these authors we can retain Bhunu and Mushayabasa [2] who in their article proposed a model on Lymphatic filariasis. They studied the local stability of the equilibrium points using as a tool the central variety theory [4] and theorem 4.1 of the paper by Castillo-Chavez and Song [5]. In addition, they considered the transition from the exposed to the infectious state as a reinfection and a simple transition, using an incidence function λ_m depending on I_m and a constant ρ. In this work, we first propose a deterministic model with a constant λ as the transition rate from the exposed state to the infectious state. Indeed, we consider that a human in the incubation phase of Lymphatic filariasis, with a low immune system, does not need to be reinfected to become infectious. For this model, we establish the global stability of the disease-free equilibrium point by using Varga's theorem [22] and the standard comparison theorem of Lakshmikantham et al [14]. From the deterministic model we reach a stochastic model by adding four white noises. We study the existence and uniqueness of the solution from results on stochastic differential equations (see [17]) and then study the extinction and persistence in mean of the stochastic system under certain assumptions. Finally, we conclude with numerical simulations to evaluate the results.

2 Deterministic model

2.1 Formulation of the deterministic model

This section is devoted to the presentation of the deterministic model. Consider N_h and N_m population size of humans and mosquitoes respectively. According to the epidemiological status of Lymphatic filariasis, we subdivide the human population into three compartments : susceptible S_h, incubating E_h and infectious I_h. Thus, the total human population is given by

$$N_h = S_h + E_h + I_h.$$

The mosquito population is subdivided into two compartments : the susceptible S_m and the infectious I_m so the total mosquito population is given by

$$N_m = S_m + I_m.$$

The human and mosquitoes beings are recruited into their corresponding susceptible populations at rates Λ_h and Λ_m, respectively. Mosquitoes experience natural death rate at a rate μ_m which is proportional to the number in each mosquito class. Similarly, human beings experience natural death at a rate μ_h, which is proportional to the number in each human-class. The mosquito ingests microfilariae when biting a human who is infected with filariasis (elephantiasis-causing nematodes) at a rate

$$\lambda_1 = \frac{\beta_h}{N_h} \text{ and } \lambda_2 = \frac{\beta_h \theta_h}{N_h}.$$

Here, β_h is the average number of mosquito bites that cause transmission of disease from infected human to susceptible mosquito; $\theta_h \in (0, 1)$ accounts for reduced number of microfilariae in the blood stream of individuals infected but not showing elephantiasis symptoms. Upon getting infected, susceptible mosquitoes enter the infected class I_m. Filariform juveniles escape from mosquito's proboscis when the insect is feeding and then penetrate wound structure of a human being at a rate

$$\lambda_m = \frac{\beta_m}{N_m},$$

where β_m is the average number of mosquito bites that cause transmission of disease from infectious mosquito to susceptible human per mosquito.
The compartmental diagram describing the progression of infection in the different compartments is given by figure 1.

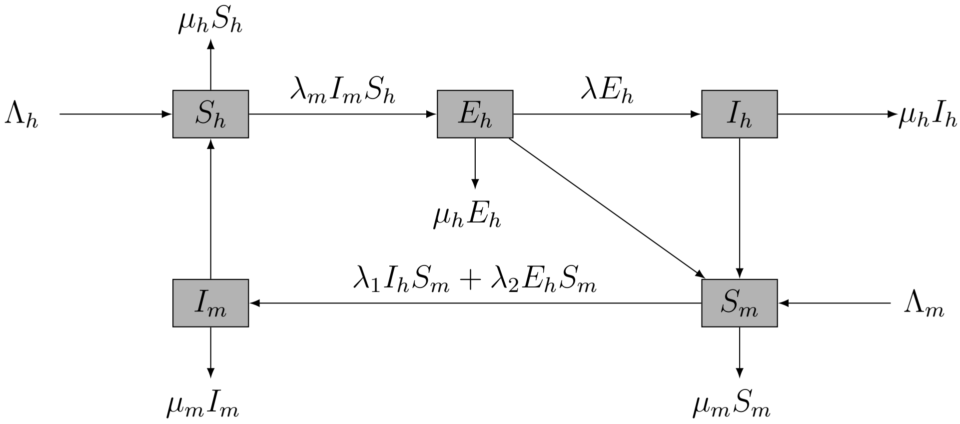

Figure 1: Transfer diagram of the deterministic model

The description leads to the following system of differential equations :

$$\begin{cases} \dfrac{dS_h(t)}{dt} = \Lambda_h - \lambda_m I_m(t) S_h(t) - \mu_h S_h(t), \\ \dfrac{dE_h(t)}{dt} = \lambda_m I_m(t) S_h(t) - (\lambda + \mu_h) E_h(t), \\ \dfrac{dI_h(t)}{dt} = \lambda E_h - \mu_h I_h(t), \\ \dfrac{dS_m(t)}{dt} = \Lambda_m - (\lambda_1 I_h(t) + \lambda_2 E_h(t)) S_m(t) \\ \qquad -\mu_m S_m(t), \\ \dfrac{dI_m(t)}{dt} = (\lambda_1 I_h(t) + \lambda_2 E_h(t)) S_m(t) - \mu_m I_m(t), \end{cases} \tag{1}$$

where

$$\begin{cases} (S_h(0), E_h(0), I_h(0), S_m(0), I_m(0)) \in \mathbb{R}_+^5, \\ \lambda_m = \dfrac{\beta_m}{N_m},\ \lambda_1 = \dfrac{\beta_h}{N_h} \text{ and } \lambda_2 = \dfrac{\beta_h \theta_h}{N_h}. \end{cases}$$

The following tables give a description of all these parameters.

Table 1: Variables for humans hosts.

Notations	**Biological description**
S_h	compartment of susceptible humans
E_h	compartment of latent humans
I_h	compartment of infectious humans

Table 2: Variables for vectors hosts.

Notations	**Biological description**
S_m	compartment of susceptible mosquitoes
I_m	compartment of infectious mosquitoes

Table 3: Parameters for the humans hosts

Parameters	Biological description
Λ_h	constant recruitment rate of human (it also includes births).
β_h	rate of passage from susceptible humans to infectious humans through blood transfusion
λ	rate of passage from latent human to infectious human
θ_h	accounts for reduced number of microfilariae in the blood stream of individuals infected but not showing elephantiasis symptoms
μ_h	mortality rate on human

Table 4: Parameters for the mosquitoes hosts

Parameters	Biological description
Λ_m	constant recruitment rate of mosquitoes
β_m	average number of mosquito bites that cause transmission of disease from infectious mosquito to susceptible human per mosquito
μ_m	Mosquito mortality rate

2.2 Existence and uniqueness of solution of the deterministic model

In this section we verify that the system (1) is mathematically and epidemiologically well-posed for all $t \geq 0$.

Proposition 2.1 *The positive orthant $\mathbb{R}^5_+$ is positively invariant.*

Proof 2.1 *Let us show that $\{S_h(t) \geq 0\}$ is positively invariant.*
Set $x = (S_h(t), E_h(t), I_h(t), S_m(t), I_m(t))^T$ where T denotes the transposition and consider the function $\varphi : \mathbb{R}^5 \longrightarrow \mathbb{R}$, $x \longmapsto -S_h$. The function φ is differentiable on $\mathbb{R}^5$ and we have
$\nabla\varphi(x) = (-1, 0, 0, 0, 0) \neq 0_{\mathbb{R}^5}$ *for all* $x \in \varphi^{-1}(\{0\})$,

$$X(x) = \begin{pmatrix} \Lambda_h \\ -(\mu_h + \lambda)E_h(t) \\ \lambda E_h(t) - \mu_h I_h(t) \\ \Lambda_m - (\lambda_1 I_h(t) + \lambda_2 E_h(t))S_m(t) - \mu_m S_m(t) \\ (\lambda_1 I_h(t) + \lambda_2 E_h(t))S_m(t) - \mu_m I_m(t) \end{pmatrix},$$

$$\langle \nabla\varphi(x), X(x) \rangle = -\Lambda_h \leq 0.$$

So the barrier theorem (see [15]) applies and therefore $\{S_h(t) \geq 0\}$ is positively invariant for system (1). Similar evidence shows that $\{E_h(t) \geq 0\}$, $\{I_h(t) \geq 0\}$, $\{S_m(t) \geq 0\}$ and $\{I_m(t) \geq 0\}$ are positively invariant. Then $\mathbb{R}^5_+$ is positively invariant for the system (1).

Lemma 2.1
The set $\Gamma_0 = \Gamma_h \times \Gamma_m \subset \mathbb{R}^3_+ \times \mathbb{R}^2_+$ where $\Gamma_h = \Big\{(S_h(t), E_h(t), I_h(t) \in \mathbb{R}^3_+ : S_h(t) + E_h(t) + I_h(t) \leq \dfrac{\Lambda_h}{\mu_h}, S_h(t) \leq \dfrac{\Lambda_h}{\mu_h}\Big\}$ and $\Gamma_m = \Big\{(S_m(t), I_m(t)) \in \mathbb{R}^2_+ : S_m(t) + I_m(t) = \dfrac{\Lambda_m}{\mu_m}, S_m(t) \leq \dfrac{\Lambda_m}{\mu_m}\Big\}$ is compact attractive positively invariant for the system (1).

Proof 2.2 *By summing the first three equations of system (1), we get :*

$$\frac{dN(t)}{dt} = \Lambda_h - \mu_h S_h(t) - \mu_h E_h(t) - \mu_h I_h(t), \ \forall t \in \mathbb{R}^+. \tag{2}$$

It follows that :

$$\frac{dN_h(t)}{dt} = \Lambda_h - \mu_h N_h(t). \qquad (3)$$

By applying the principle of solving differential inequalities developed by Birkhoff and Rota [3] to inequality (3), *we get :*

$$N_h(t) = \frac{\Lambda_h}{\mu_h} - (N_h(0) - \frac{\Lambda_h}{\mu_h}) \exp(-\mu_h t). \quad (4)$$

By passage to the limit in (4), *we obtain :*

$$\lim_{t \to +\infty} N_h(t) = \frac{\Lambda_h}{\mu_h}.$$

By the same principle, we show that :

$$\lim_{t \to +\infty} N_m(t) = \frac{\Lambda_m}{\mu_m}.$$

Therefore Γ_0 *is positively invariant.*

Thus, without loss of generality we assume that

$$N_h(t) = \frac{\Lambda_h}{\mu_h} \text{ et } N_m(t) = \frac{\Lambda_m}{\mu_m}, \forall t \in \mathbb{R}^+.$$

2.3 Existence of equilibria

In this section, we calculate the basic reproduction number $\mathcal{R}_0$ and equilibria points of system (1). System (1) have a disease-free equilibrium given by

$$\mathcal{E}_0 = (S_h^0, E_h^0, I_h^0, S_m^0, I_m^0) \text{ where } S_h^0 = \frac{\Lambda_h}{\mu_h},$$
$$E_h^0 = 0, I_h^0 = 0, S_m^0 = \frac{\Lambda_m}{\mu_m} \text{ and } I_m^0 = 0.$$

By using "next generation matrix" method proposed by van den Driessche and Watmough [10], we arrive to calculate $\mathcal{R}_0$.

Proposition 2.2 *The basic reproduction number of system* (1) *is*

$$\mathcal{R}_0 = \sqrt{\frac{\Lambda_h \Lambda_m \lambda_m (\lambda_1 \mu_h + \lambda \lambda_2)}{\mu_h^2 \mu_m^2 (\lambda + \mu_h)}}.$$

Proof 2.3 *The new infection matrix F and the transition matrix V are given by*

$$\mathrm{F} = \begin{pmatrix} 0 & 0 & \lambda_m S_h^0(t) \\ 0 & 0 & 0 \\ \lambda_2 S_m^0 & \lambda_1 S_m^0 & 0 \end{pmatrix} \text{ and}$$
$$\mathrm{V} = \begin{pmatrix} -(\lambda + \mu_m) & 0 & 0 \\ \lambda & -\mu_h & 0 \\ 0 & 0 & -\mu_m \end{pmatrix}.$$

So the next generation matrix is

$$-\mathrm{FV}^{-1} = \begin{pmatrix} 0 & 0 & \frac{\lambda_m}{\mu_m} S_h^0 \\ 0 & 0 & 0 \\ \frac{\lambda_2 S_m^0}{\lambda + \mu_h} + \frac{\lambda \lambda_1 S_m^0}{\mu_h(\lambda + \mu_h)} & \frac{\lambda_2}{\mu_h} & 0 \end{pmatrix}.$$

Therefore

$$\begin{aligned} \mathcal{R}_0 &= \rho(-FV^{-1}) \qquad (5) \\ &= \sqrt{\frac{S_h^0 S_m^0 \lambda_m (\lambda_2 \mu_h + \lambda \lambda_1)}{\mu_h \mu_m (\lambda + \mu_h)}}. \end{aligned}$$

Replacing S_h^0 *and* S_m^0 *by* (5) *we get*

$$\mathcal{R}_0 = \sqrt{\frac{\Lambda_h \Lambda_m \lambda_m (\lambda_2 \mu_h + \lambda \lambda_1)}{\mu_h^2 \mu_m^2 (\lambda + \mu_h)}}.$$

2.4 Stability study of disease-free equilibrium

In this section, we study the stability of of disease-free equilibrium of system (1). The local stability result is stated as follow.

Theorem 2.1 *If* $\mathcal{R}_0 < 1$, *then the disease-free equilibrium point is locally asymptotically stable.*

Proof 2.4 *Let us suppose that* $\mathcal{R}_0 < 1$. *It is sufficient to show that the eigenvalues of the Jacobian matrix of system* (1) *at the point* $\mathcal{E}_0$ *are negative real part. By a simple calculation, we show that the characteristic equation at the point* $\mathcal{E}_0$ *checks.*

$$(\mu_h + X)(\mu_m + X) Q(X) = 0,$$

where,

$$\begin{aligned} & Q(X) \\ = & X^3 + \left[(\mu_h + \lambda) + (\mu_h + \mu_m)\right] X^2 + \\ & \left[(\mu_h + \lambda)(\mu_h + \mu_m) + \mu_m \mu_h\right] X \\ & -\lambda_2 \lambda_m S_h^0 S_m^0 X + \mu_m \mu_h (\mu_h + \lambda) \\ & -\mu_h h \lambda_2 \lambda_m S_h^0 S_m^0 - \lambda_m \lambda \lambda_1 S_h^0 S_m^0. \end{aligned}$$

Let

$$\begin{aligned} \sigma_1 &= (\mu_h + \lambda) + (\mu_h + \mu_m), \\ \sigma_2 &= (\mu_h + \lambda)(\mu_h + \mu_m) + \\ & \quad \mu_m \mu_h - \lambda_2 \lambda_m S_h^0 S_m^0, \\ \sigma_3 &= \mu_m \mu_h (\mu_h + \lambda) - \mu_h \lambda_2 \lambda_m S_h^0 S_m^0 \\ & \quad -\lambda_m \lambda \lambda_1 S_h^0 S_m^0. \end{aligned}$$

We have

$$\sigma_3 = \mu_m \mu_h (\mu_h + \lambda)\left[1 - \mathcal{R}_0^2\right].$$

Using the fact that $\mathcal{R}_0 < 1$, we get

$$\begin{aligned} & S_h^0 S_m^0 \lambda_m (\lambda_2 \mu_h + \lambda \lambda_1) < \mu_h \mu_m (\lambda + \mu_h) \\ \Rightarrow \ & S_h^0 S_m^0 \lambda_m \lambda_2 \mu_h < \mu_h \mu_m (\lambda + \mu_h) \\ \Rightarrow \ & S_h^0 S_m^0 \lambda_m \lambda_2 < \mu_m (\lambda + \mu_h) \\ \Rightarrow \ & -S_h^0 S_m^0 \lambda_m \lambda_2 > -\mu_m (\lambda + \mu_h) \\ \Rightarrow \ & \sigma_2 > (\mu_h + \lambda)(\mu_h + \mu_m) \\ & +\mu_m \mu_h - \mu_m (\lambda + \mu_h) \\ \Rightarrow \ & \sigma_2 > \mu_h (\mu_h + \lambda). \end{aligned}$$

Thus $\sigma_1, \sigma_2, \sigma_3 > 0$ for $\mathcal{R}_0 < 1$. In addition,

$$\begin{aligned} & \sigma_1 \sigma_2 > \mu_h \mu_m (\lambda + \mu_h) \Rightarrow \sigma_1 \sigma_2 > \\ & \mu_h \mu_m (\lambda + \mu_h)(1 - \mathcal{R}_0) = \sigma_3. \end{aligned}$$

So by applying the Routh Hurtwiz criterion, we conclude that all roots of Q are negative real part. Moreover the eigenvalues $-\mu_h$ and $-\mu_m$ are negative. This completes the proof.

Let us introduce the notations

$$\liminf_{t \to \infty} \varphi(t) = \varphi_\infty,$$
$$\limsup_{t \to \infty} \varphi(t) = \varphi^\infty.$$

To establish the global stability of the equilibrium point $\mathcal{E}_0$, let us begin by stating the following lemma.

Lemma 2.2 *(see [12]) Let $\varphi : \mathbb{R}^+ \to \mathbb{R}$ be a bounded and continuously differentiable function. Then, there are two sequences $\{v_n\}$ and $\{t_n\}$ such that $v_n \longrightarrow \infty$, $t_n \longrightarrow \infty$, $\varphi(v_n) \longrightarrow \varphi_\infty$, $\varphi'(v_n) \longrightarrow 0$ and $\varphi(t_n) \longrightarrow \varphi^\infty$ and $\varphi'(t_n) \longrightarrow 0$ when $n \longrightarrow \infty$.*

Theorem 2.2 *The equilibrium point $\mathcal{E}_0$ of the model (1) is globally asymptotically stable when $\mathcal{R}_0 < 1$.*

Proof 2.5 *Let $(S_h(t), E_h(t), I_h(t), S_m(t), I_m(t))$ be a solution of the system with initial conditions in Γ_0. According to the lemma 2.1, Γ_0 is positively invariant and therefore we deduce that $S_h^\infty \leq \dfrac{\Lambda_h}{\mu_h}$ and $S_m^\infty \leq \dfrac{\Lambda_m}{\mu_m}$. Using the lemma 2.1, we obtain the following system of linear inequalities*

$$\begin{cases} \dfrac{dE_h(t)}{dt} \leq \dfrac{\Lambda_h}{\mu_h} \lambda_m I_m(t) - (\lambda + \mu_h) E_h(t), \\ \dfrac{dI_h(t)}{dt} \leq \lambda E_h - \mu_h I_h(t), \\ \dfrac{dI_m(t)}{dt} \leq \dfrac{\Lambda_m}{\mu_m} (\lambda_1 I_h(t) + \lambda_2 E_h(t)) - \mu_m I_m(t). \end{cases} \tag{6}$$

System (6) can be written as

$$dY(t) \leq AY, \tag{7}$$

where

$$Y(t) = \begin{pmatrix} E_h(t) \\ I_h(t) \\ I_m(t) \end{pmatrix} \text{ and}$$

$$A = \begin{pmatrix} -(\lambda + \mu_h) & 0 & \dfrac{\Lambda_h}{\mu_h} \lambda_m \\ \lambda & -\mu_h & 0 \\ \dfrac{\Lambda_m}{\mu_m} \lambda_2 & \dfrac{\Lambda_m}{\mu_m} \lambda_1 & -\mu_m \end{pmatrix}$$

The matrix A is a Metzler matrix and its linear decomposition is given by : $A = F + V$, where

$$F = \begin{pmatrix} 0 & 0 & \frac{\Lambda_h}{\mu_h}\lambda_m \\ 0 & 0 & 0 \\ \frac{\Lambda_m}{\mu_m}\lambda_2 & \frac{\Lambda_m}{\mu_m}\lambda_1 & 0 \end{pmatrix} \text{ and}$$

$$V = \begin{pmatrix} -(\lambda+\mu_h) & 0 & 0 \\ \lambda & -\mu_h & 0 \\ 0 & 0 & -\mu_m \end{pmatrix}.$$

Moreover V is a Metzler matrix and therefore inverse invertible

$$V^{-1} = \begin{pmatrix} -\frac{1}{\lambda+\mu_h} & 0 & 0 \\ -\frac{\lambda}{\mu_h(\lambda+\mu_h)} & -\frac{1}{\mu_h} & 0 \\ 0 & 0 & -\frac{1}{\mu_m} \end{pmatrix}.$$

We also note that $F \geq 0$, $-V^{-1} \geq 0$ and $\mathcal{R}_0 = \rho(-FV^{-1})$. Since $\mathcal{R}_0 = \rho(-FV^{-1}) < 1$, then the theorem of Varga (see [22]) allows us to deduce that A is asymptotically stable, i.e. the eigenvalues of A are negative real part. Using the standard comparison theorem of Laksmikantham et al. see [14], we deduce that $E_h(t) \longrightarrow 0, I_h(t) \longrightarrow 0, I_m \longrightarrow 0$ when $t \longrightarrow \infty$ and therefore $E_h^\infty = 0$, $I_h^\infty = 0$ and $I_m^\infty = 0$. According to the lemma 2.2, there exists a sequence $\{t_n\}$ such that $t_n \longrightarrow \infty$, $S_h(t_n) \longrightarrow S_{h,\infty}$, $S_m(t_n) \longrightarrow S_{m,\infty}$ and $S_h'(t_n) \longrightarrow 0$, $S_m'(t_n) \longrightarrow 0$ when $n \longrightarrow 0$. Thus,

$$\begin{aligned} \frac{dS_h(t_n)}{dt} &= \Lambda_h - \lambda_m I_m(t_n)S_h(t_n) \\ &- \mu_h S_h(t_n), \\ \frac{dS_m(t_n)}{dt} &= \Lambda_m - \lambda_1 I_h(t_n)S_m(t_n) \\ &- \lambda_2 E_h(t_n)S_m(t_n) - \mu_m S_m(t_n). \end{aligned}$$

When $n \longrightarrow \infty$, then

$$\begin{aligned} 0 &= \Lambda_h - \lambda_m I_{m,\infty}S_{h,\infty} - \mu_h S_{h,\infty}, \\ 0 &= \Lambda_m - (\lambda_1 I_{h,\infty} + \lambda_2 E_{h,\infty})S_{m,\infty} \\ &- \mu_m S_{m,\infty}. \end{aligned}$$

It follows that

$$\begin{aligned} 0 &\geq \Lambda_h - \lambda_m I_m^\infty S_{h,\infty} - \mu_h S_{h,\infty}, \\ 0 &\geq \Lambda_m - (\lambda_1 I_h^\infty + \lambda_2 E_h^\infty)S_{m,\infty} \\ &- \mu_m S_{m,\infty}. \end{aligned}$$

Thus

$$\begin{aligned} \frac{\Lambda_h}{\mu_h} \leq S_{h,\infty} \leq S_h^\infty \leq \frac{\Lambda_h}{\mu_h}, \\ \frac{\Lambda_m}{\mu_m} \leq S_{m,\infty} \leq S_m^\infty \leq \frac{\Lambda_m}{\mu_m}. \end{aligned}$$

Thanks to the fact that $E_h^\infty = I_h^\infty = I_m^\infty = 0$ and $S_h^\infty \leq \frac{\Lambda_h}{\mu_h}$, $S_m^\infty \leq \frac{\Lambda_m}{\mu_m}$. It follows that $\lim_{t\to\infty} S_h(t) = \frac{\Lambda_h}{\mu_h}$ and $\lim_{t\to\infty} S_m(t) = \frac{\Lambda_m}{\mu_m}$. So $(S_h(t), E_h(t), I_h(t), S_m(t), I_m(t)) \rightarrow \mathcal{E}_0$ in Γ_0 when $t \rightarrow \infty$. This completes the proof.

Let us now introduce the proportions

$$\begin{aligned} s_h(t) &= \frac{S_h(t)}{N_h},\ e_h(t) = \frac{E_h(t)}{N_h}, \\ i_h(t) &= \frac{I_h(t)}{N_h},\ s_m(t) = \frac{S_m(t)}{N_m}, \\ i_m(t) &= \frac{I_m(t)}{N_m}, \end{aligned}$$

and also, taking into account the equalities $s_h(t)+e_h(t)+i_h(t) = 1$ and $s_m(t)+i_m(t) = 1$. Thus system (1) reduces to

$$\begin{cases} \frac{ds_h(t)}{dt} = \mu_h - \beta_1(t)i_m(t)s_h(t) - \mu_h s_h(t), \\ \frac{de_h(t)}{dt} = \beta_1 i_m(t)s_h(t) - (\lambda+\mu_h)e_h(t), \\ \frac{di_h(t)}{dt} = \lambda e_h - \mu_h i_h(t), \\ \frac{ds_m(t)}{dt} = \mu_m - (\beta_3 i_h(t) + \beta_2 e_h(t))s_m(t) \\ \qquad -\mu_m s_m(t), \\ \frac{di_m(t)}{dt} = (\beta_3 i_h(t) + \beta_2 e_h(t))s_m(t) - \mu_m i_m(t), \end{cases} \tag{8}$$

where,

$$\beta_1 = \beta_m,\ \beta_2 = \theta_h\beta_h,\ \beta_3 = \beta_h.$$

3 Stochastic model

We give some basic theory in equations differentials stochastic (see [16]). Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space with $(\mathcal{F}_t)_{t\geq 0}$ filtration that satisfying the usual

conditions. We note
$\mathbb{R}^5_+ = \Big\{ (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 :$
$x_1 > 0, x_2 > 0, x_3 > 0, x_4 > 0, x_5 > 0 \Big\}$ and
$\chi^+ = \Big\{ (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5_+ : \ x_1 + x_2 + x_3 < 1, x_4 + x_5 < 1 \Big\}$.

The following stochastic system is considered

$$dX(t) = f(t, X(t))dt + g(t, X(t))dB(t), \quad (9)$$

for $t \geq t_0$ with$X(t_0) = X_0 \in \mathbb{R}^n$, $B(t)$ denotes n dimensional standard Brownian motion defined on the above probability space. Define the differential operator $\mathcal{L}$ associated to (9) by :

$$\begin{aligned} \mathcal{L}V(t,X) &= \frac{\partial V(t,X)}{dt} + f^T \frac{\partial V(t,X)}{dX} \quad (10) \\ &+ \frac{1}{2} Tr \left[g^T \frac{\partial^2 V(t,X)}{dX^2} g \right] \text{ where} \\ & V(t,X) \in \mathcal{C}^{1,2} \left(\mathbb{R} \times \mathbb{R}^m \right). \end{aligned}$$

The stochastic version of the deterministic system (8) is given by :

$$\begin{cases} ds_h(t) = \left[\mu_h - \beta_1 i_m(t) s_h(t) - \mu_h s_h(t) \right] dt + \\ \qquad \mathcal{P}_1(t), \\ de_h(t) = \left[\beta_1 i_m(t) s_h(t) - (\lambda + \mu_h) e_h(t) \right] dt + \\ \qquad \mathcal{P}_2(t), \\ di_h(t) = \left[\lambda e_h - \mu_h i_h(t) \right] dt + \eta_2 e_h(t) dB_2(t), \\ ds_m(t) = \left[\mu_m - (\beta_3 i_h(t) + \beta_2 e_h(t)) s_m(t) \right] dt \\ \qquad -\mu_m s_m(t) dt + \mathcal{P}_3(t), \\ di_m(t) = \left[(\beta_3 i_h(t) + \beta_2 e_h(t)) s_m(t) \right] dt \\ \qquad -\mu_m i_m(t) dt + \mathcal{P}_4(t) \end{cases} \quad (11)$$

where B_j,$j = \overline{1,4}$ are mutually independent Brownians and η_j,$j = \overline{1,4}$ are their respective intensities and

$$\begin{cases} \mathcal{P}_1(t) = -\eta_1 i_m(t) s_h(t) dB_1(t); \\ \mathcal{P}_2(t) = \eta_1 i_m(t) s_h(t) dB_1(t) \\ \qquad -\eta_2 e_h(t) dB_2(t); \\ \mathcal{P}_3(t) = -\eta_3 i_h(t) s_m(t) dB_3(t) \\ \qquad -\eta_4 e_h s_m(t) dB_4(t); \\ \mathcal{P}_4(t) = \eta_3 i_h(t) s_m(t) dB_3(t) \\ \qquad +\eta_4 e_h s_m(t) dB_4(t). \end{cases}$$

3.1 Existence of a positive global solution

In this section, inspired by the method in [8, 18], we show that there is a unique global positive solution of system (11). We establish the following theorem.

Theorem 3.1 *For all initial values* $x(0) = (s_h(0), e_h(0), i_h(0), s_m(0), i_m(0)) \in \chi^+$, *there is a unique solution* $x(t) = (S_h(t), E_h(t), I_h(t), s_m(t), i_m(t))$ *for the system such that*
$\mathbb{P}(x(t) \in \chi^+) = 1$ *for all* $t \geq 0$.

Proof 3.1 *Let's call it* $n_h(t) = S_h(t) + E_h(t) + i_h(t)$ *the sum of the respective proportions of susceptible, infected and recovered humans at time* t *and* $n_m(t) = s_m(t) + i_m(t)$ *that of the proportions of susceptible and infected mosquitoes at time* t. *For all* $x(\xi) = (s_h(\xi), e_h(\xi), i_h(\xi), s_m(\xi), i_m(\xi))$ *belongs to* $\mathbb{R}^5_+$ *a.s we have :*

$$\begin{aligned} dn_h(\xi) &= \left[\mu_H - \mu_h n_h(\xi) \right] d\xi \ a.s., (12) \\ dn_m(\xi) &= \left[\mu_m - \mu_m n_m(\xi) \right] d\xi \ a.s. (13) \end{aligned}$$

Using Gronwall's lemma, we get :

$$\begin{aligned} n_h(\xi) &= 1 + (n_h(0) - 1) \exp(-\mu_h \xi) \ a.s., \\ n_m(\xi) &= 1 + (n_m(0) - 1) \exp(-\mu_m \xi) \ a.s. \end{aligned}$$

Since $(s_h(0), e_h(0), i_h(0), s_m(0), i_m(0)) \in \chi^+$, *then* $n_h(\xi) < 1$ *a.s. and* $n_m(\xi) < 1$ *a.s. So,* $x(\xi) \in (0,1)^5$ *for all* $\xi \in [0,t]$. *Moreover, since the coefficients of system* (11) *are locally Lipschitzian, there is a unique solution* $(S_h(t), E_h(t), I_h(t), s_m(t), i_m(t))$ *on any fixed interval* $[0,t]$.
Let $x(t) = (S_h(t), E_h(t), I_h(t), s_m(t), i_m(t))$ *a solution of system* (11) *where,* $t \in [0, \tau_e)$ *and* τ_e *is the explosion time. To show that* $x(t)$ *is global, we need only show that* $\tau_e = \infty$. *To this end, let* $\alpha_0 \in (0,1)$ *be sufficiently small such that* $(s_h(0), e_h(0), i_h(0), s_m(0), i_m(0)) \in (\alpha_0, 1 - \alpha_0)^5$. *For* $0 < \alpha \leq \alpha_0$, *define the stopping time* τ_α *through*

$$\begin{aligned} \tau_\alpha = \inf \Big\{ t \in [0, \tau_e) : \\ s_h(t) \wedge e_h(t) \wedge i_h(t) \wedge s_m(t) \wedge i_m(t) \leq \alpha \ or \\ s_h(t) \vee e_h(t) \vee i_h(t) \vee s_m(t) \vee i_m(t) \geq 1 - \alpha \Big\} \end{aligned}$$

where in this paper we assume that $\inf(\emptyset) = \infty$. *Clearly* τ_α *is increasing as* $\alpha \to 0$. *Set* $\tau_\infty = \lim_{\alpha\to 0}\tau_\alpha$, *hence* $\tau_\infty \leq \tau_e$ *because* $\tau_\alpha \leq \tau_e$. *If we can verify that* $\tau_\infty = \infty$ *a.s, then* $\tau_e = \infty$ *and* $x(t) = (s_h(t), e_h(t), i_h(t), s_m(t), i_m(t)) \in \chi^+$, $\forall t \geq 0$. *If this assertion is not true, then there exists a pair constant* $T > 0$ *and* $\zeta \in (0,1)$ *such that*

$$\mathbb{P}(\{\tau_\infty \leq T\}) > \zeta. \tag{14}$$

Therefore, there is an positive constant $\alpha_1 \leq \alpha_0$ *such that*

$$\mathbb{P}(\{\tau_\alpha \leq T\}) > \zeta, \ \forall\alpha \in (0, \alpha_1]. \tag{15}$$

Define a non-negative $\mathcal{C}^2-$*function* w *of* $\in \chi^+$ *in* $\mathbb{R}^+$ *as follows*

$$w(x(t)) = w_{s_h} + w_{e_h} + w_{i_h} + w_{s_m} + w_{i_m},$$

where,

$$\begin{aligned} w_{s_h} &= -\ln(s_h(t)),\\ w_{e_h} &= -\ln(e_h(t)),\\ w_{i_h} &= -\ln(i_h(t)),\\ w_{s_m} &= -\ln(s_m(t)),\\ w_{i_m} &= -\ln(i_m(t)). \end{aligned}$$

Let $\alpha \in (0, \alpha_0]$ *and let* $T > 0$ *be arbitrary. Using Itô's formula [17] and for all* $t \geq 0$ *fixed and* $\xi \in [0, t]$, *we get*

$$\begin{aligned} dw_{s_h} &= \left[\frac{-\mu_h}{s_h(\xi)} + \beta_1 i_m(\xi) + \mu_h\right] d\xi \\ &\quad + \frac{1}{2}\eta_1^2 i_m^2(\xi) d\xi + \eta_1 i_m(\xi) dB_1(\xi) \\ &\leq [\beta_1 i_m(\xi) + \mu_h]\, d\xi \\ &\quad + \frac{1}{2}\eta_1^2 i_m^2(\xi) d\xi + \eta_1 i_m(\xi) dB_1(\xi) \\ &\leq \left(\beta_1 + \mu_h + \frac{1}{2}\eta_1^2\right) d\xi \\ &\quad + \eta_1 i_m(\xi) dB_1(\xi), \end{aligned}$$

thanks to the fact that $i_m \in (0,1)$.

$$\begin{aligned} dw_{e_h} &= \left[-\beta_1 i_m(\xi) s_h(\xi) e_h^{-1}(\xi) + (\lambda + \mu_h)\right] d\xi \\ &\quad + \left[\frac{1}{2}\eta_1^2 i_m^2(\xi) s_h^2(\xi) e_h^{-2}(\xi) + \frac{1}{2}\eta_2^2\right] d\xi \\ &\quad - \eta_1 i_m(\xi) s_h(\xi) e_h^{-1}(\xi) dB_1(\xi) \\ &\quad + \eta_2 dB_2(\xi) \\ &\leq \left[(\lambda + \mu_h + \frac{1}{2}\eta_2^2) + \frac{1}{2}\eta_1 e_h^{-2}(\xi)\right] d\xi \\ &\quad - \eta_1 i_m(\xi) s_h(\xi) e_h^{-1}(\xi) dB_1(\xi) \\ &\quad + \eta_2 dB_2(\xi) \end{aligned}$$

thanks to the fact that $i_m, s_h \in (0,1)$.

$$\begin{aligned} dw_{i_h} &= \left[-\lambda e_h(\xi) i_h^{-1}(\xi)\right] d\xi \\ &\quad + \left[\mu_h + \frac{1}{2}\eta_2^2 e_h^2(\xi) i_h^{-2}(\xi)\right] d\xi \\ &\quad - \eta_1 e_h(\xi) i_h^{-1}(\xi) dB_2(\xi) \\ &\leq \left(\mu_h + \frac{1}{2}\eta_2^2 i_h^{-2}(\xi)\right) d\xi \\ &\quad - \eta_2 e_h(\xi) i_h^{-1}(\xi) dB_2(\xi) \end{aligned}$$

since $e_h \in (0,1)$. *As the applications* e_h *and* i_h *are continuous then by using the theorem of Weierstrass, we obtain*

$$\begin{aligned} &\inf_{\xi\in[0,\max\{T,\tau_\alpha\}]} e_h(\xi) = c_1 < \infty \text{ and} \\ &\inf_{\xi\in[0,\max\{T,\tau_\alpha\}]} i_h(\xi) = c_2 < \infty. \end{aligned}$$

So,

$$\begin{aligned} dw_{s_h} + dw_{e_h} + dw_{i_h} &\leq k_1(\xi) + g_1(\xi) dB_1(\xi) \\ &\quad + g_2(\xi) dB_2(\xi), \end{aligned}$$

where

$$\begin{aligned} k_1 d\xi &= (\beta_1 + 3\mu_h + \lambda)\, d\xi \\ &\quad + \frac{1}{2}\left(\eta_1^2 + \eta_2^2 + \eta_1^2 c_1^{-2} + \eta_2^2 c_2^{-2}\right) d\xi, \\ g_1(\xi) &= \left(\eta_1 i_m(\xi) - \eta_1 i_m(\xi) s_h(\xi) e_h^{-1}(\xi)\right), \\ g_2(\xi) &= \left(\eta_2 dB_2(\xi) - \eta_2 e_h(\xi) i_h^{-1}(\xi)\right). \end{aligned} \tag{16}$$

Likewise by Itô's formula, we get

$$\begin{aligned} dw_{s_m} &= \left[\frac{-\mu_m}{s_m} + (\beta_3 i_h(\xi) + \beta_2 e_h(\xi))\right] d\xi \\ &\quad + \left[\mu_m + \frac{1}{2}\eta_3^2 i_h^2(\xi) + \frac{1}{2}\eta_4^2 e_h^2(\xi)\right] d\xi \\ &\quad + \eta_3 i_h(\xi) dB_3(\xi) + \eta_4 e_h(\xi) dB_4(\xi) \\ &\leq \left(\beta_3 + \beta_2 + \mu_m + \frac{1}{2}\eta_3^2 + \frac{1}{2}\eta_4^2\right) d\xi \\ &\quad + \eta_3 i_h(\xi) dB_3(\xi) + \eta_4 e_h(\xi) dB_4(\xi), \end{aligned}$$

and

$$\begin{aligned} dw_{i_m} &= \left[-(\beta_3 i_h(\xi)+\beta_2 e_h(\xi))s_m(\xi)i_m^{-1}\right]d\xi \\ &+\left[\mu_m+\frac{1}{2}\eta_3^2 i_h^2(\xi)s_m^2(\xi)i_m^{-2}(\xi)\right]d\xi \\ &+\frac{1}{2}\eta_4^2 e_h^2 s_m^2(\xi)(\xi)i_m^{-2}(\xi)d\xi \\ &-\eta_3 i_h(\xi)i_m^{-1}(\xi)dB_3(\xi) \\ &-\eta_4 e_h(\xi)i_m^{-1}(\xi)dB_4(\xi) \\ &\leq \left[\mu_m+\frac{1}{2}\eta_3^2 i_m^{-2}(\xi)+\frac{1}{2}\eta_4^2 i_m^{-2}(\xi)\right]d\xi \\ &-\eta_3 i_h(\xi)i_m^{-1}(\xi)dB_3(\xi) \\ &-\eta_4 e_h(\xi)i_m^{-1}(\xi)dB_4(\xi). \end{aligned}$$

Since the applications i_m is continuous then by using the theorem of Weierstrass, we obtain

$$\inf_{\xi\in[0,\max\{T,\tau_\alpha\}]} i_m(\xi)=c_3<\infty.$$

So,

$$\begin{aligned} dw_{s_m}+dw_{i_m} &\leq k_2(\xi)+g_3(\xi)dB_3(\xi) \\ &+g_4(\xi)dB_4(\xi), \end{aligned}$$

where,

$$\begin{aligned} k_2 d\xi &= \frac{1}{2}\left(\eta_3^2+\eta_4^2+\eta_3^2 c_3^{-2}+\eta_4^2 c_3^{-2}\right)d\xi \\ &+(\beta_3+\beta_2+2\mu_m)\,d\xi \\ g_3(\xi) &= \left(\eta_3 i_h(\xi)-\eta_3 i_h(\xi)i_m^{-1}(\xi)\right), \\ g_4(\xi) &= \left(\eta_4 e_h(\xi)-\eta_4 e_h(\xi)i_m^{-1}(\xi)\right). \end{aligned}$$

Therefore, we get

$$dw(x(\xi))\leq(k_1+k_2)\,d\xi+\sum_{j=1}^{4}g_j(\xi)dB_j(\xi). \quad (17)$$

Integrating above inequality from 0 to $\tau_\alpha\wedge T$ on both sides yields

$$\begin{aligned} w(x(\tau_\alpha\wedge T)) &\leq w(x(0)) \\ &+(k_1+k_2)(\tau_\alpha\wedge T) \\ &+\sum_{j=1}^{4}\int_0^{\tau_\alpha\wedge T}g_j(\xi)dB_j(\xi). \end{aligned}$$

Let

$$h_j=\sup_{\xi\in[0,\tau_\alpha\wedge T]}\{g_j(\xi)\},\ j=\overline{1,4}. \quad (18)$$

Since the applications e_h, i_h and i_m are continuous then by using Weierstrass theorem, we get

$$h_j<\infty,\ j=\overline{1,4}. \quad (19)$$

Hence

$$\begin{aligned} w(x(\tau_\alpha\wedge T)) &\leq w(x(0))+(k_1+k_2)(\tau_\alpha\wedge T) \\ &+\textstyle\sum_{j=1}^{4}h_jB_j(t). \end{aligned}$$

By taking the expectations on both sides, we get

$$\begin{aligned} \mathbb{E}\left(w(x(\tau_\alpha\wedge T))\right) &\leq w(x(0)) \\ &+ (k_1+k_2)\,\mathbb{E}\left(\tau_\alpha\wedge T\right) \\ &\leq w(x(0))+(k_1+k_2)\,T. \end{aligned}$$

Set $\Omega_\alpha=\{\tau_\alpha\leq T\}$ for $0<\alpha\leq\alpha_1$. From (15) we have

$$\mathbb{P}(\Omega_\alpha)>\zeta. \quad (20)$$

Noticing that every $\omega\in\Omega_\alpha$ some components $x(\tau_\alpha)$ equals α or $1-\alpha$. Thus,

$$w(x(\tau_\alpha))\geq[-\ln\alpha]\wedge[\ln(1-\alpha)].$$

Furthermore, we have

$$\begin{aligned} &\mathbb{E}\left[w(x(\tau_\alpha\wedge T,\omega))\right] \\ =\ &\mathbb{E}\left[1_{\Omega_\alpha}(\omega)w(x(\tau_\alpha\wedge T,\omega))\right] \\ &+\mathbb{E}\left[1_{\overline{\Omega}_\alpha}(\omega)w(x(\tau_\alpha\wedge T,\omega))\right] \\ \geq\ &\mathbb{E}\left[1_{\Omega_\alpha}(\omega)w(x(\tau_\alpha\wedge T,\omega))\right] \\ \geq\ &\mathbb{E}\left[1_{\Omega_\alpha}(\omega)w(x(\tau_\alpha,\omega))\right] \\ \geq\ &\mathbb{P}\left(\Omega_\alpha\right)[-\ln\alpha]\wedge[\ln(1-\alpha)] \\ \geq\ &\zeta\,[-\ln\alpha]\wedge[\ln(1-\alpha)], \end{aligned}$$

where 1_{Ω_α} denotes the indicator function of Ω_α and $\overline{\Omega}_\alpha=\{\tau_\alpha>T\}$ for $0<\alpha\leq\alpha_1$. Consequently

$$w(x(0))+(k_1+k_2)\,T\geq \quad (21)$$

$$\zeta\,[-\ln\alpha]\wedge[\ln(1-\alpha)]. \quad (22)$$

Letting $\alpha\longrightarrow 0$ in (22) leads to

$$\infty>w(x(0))+(k_1+k_2)\,T=\infty.a.s., \quad (23)$$

which yields the contradiction. Hence we derive $\tau_\infty=\infty$ .a.s. This means that the solutions $x(t)$ will not explode in a finite time a.s. This completes the proof.

3.2 Extinction of filariasis

In the deterministic model, the value of $\mathcal{R}_0$ guarantees the persistence or extinction of filariasis. When $\mathcal{R}_0$ is less than one, the disease-free equilibrium point of the system is globally asymptotically stable. This means that the number of cases of filariasis infection decreases until it becomes zero. In this section, we determine some criteria for the extinction of filariasis in the case of the stochastic model. Set

$$\begin{aligned}\mathcal{R}_0^e &= \frac{\beta_1^2\eta_1^{-2}}{2\left(\eta_2^2+2\lambda+2\mu_h\right)},\\ \mathcal{R}_0^m &= \frac{1}{\mu_m}\left(\frac{\beta_3^2\eta_4^2+\beta_2^2\eta_3^2}{\eta_3^2\eta_4^2}\right),\\ \mathcal{R}_0^i &= \frac{\mu_h^{-2}\lambda^2}{\eta_2^2} \text{ and } \mathcal{R}_0^{Ex}=\mathcal{R}_0^e\vee\mathcal{R}_0^i\vee\mathcal{R}_0^m.\end{aligned}$$

Theorem 3.2 *For any solution* $x(.) = ((s_h(.), e_h(.), i_h(.), s_m(.), i_m(.))$ *with initial condition*
$x(0) = ((s_h(0), e_h(0), i_h(0), s_m(0), i_m(0)) \in \chi^+$ *holds*

$$\begin{aligned}\limsup_{t\to\infty}\frac{\ln(e_h(t))}{t} &< \frac{1}{2}\left(\eta_2^2+2\lambda+2\mu_h\right)\times\\ &\quad\left(\mathcal{R}_0^e-1\right) a.s.,\\ \limsup_{t\to\infty}\frac{\ln(i_h(t))}{t} &< \mu_h\left(\mathcal{R}_0^i-1\right) a.s.,\\ \limsup_{t\to\infty}\frac{\ln(i_m(t))}{t} &< \mu_m\left(\mathcal{R}_0^m-1\right) a.s.\end{aligned}$$

Thus if

$$\mathcal{R}_0^{Ex}<1, \tag{24}$$

hold then $x(.)$ *converges exponentially almost surly to the equilibrium point* $(1, 0, 0, 1, 0)$ *with probability one.*

To verify theorem 3.2, we first present some lemmas which will be used later.

Lemma 3.1 *[16] Let* $g = (g_1, g_2, ..., g_m) \in \mathcal{L}^2\left(\mathbb{R}_+, \mathbb{R}^{1\times m}\right)$, *and let* T, ϱ *and* θ *be any positive numbers. Then*

$$\mathcal{P}\Big\{\sup_{0\le t\le T}\left[\int_0^t g(u)dB(u)-\frac{\varrho}{2}\int_0^t \mid g(u)\mid^2 du\right] > \theta\Big\} \le exp\left(-\varrho\theta\right).$$

Lemma 3.2 *[17] Let* $\{A_t\}_{t\ge0}$ *and* $\{U_t\}_{t\ge0}$ *two increasing continuous processes and adapted with* $A_0 = U_0 = 0$ *a.s. Let* $\{M_t\}_{t\ge0}$ *a local continuous real-valued martingale with* $M_0 = 0$ *a.s. Let* ξ *a non-negative variable and* $\mathcal{F}_0-$*mesurable. Define*

$$X_t = \xi + A_t - U_t + M_t,\ \textit{for } t \ge 0.$$

If X_t *is non-negative, then*

$$\begin{aligned}&\left\{\lim_{t\to+\infty} A_t<\infty\right\}\subset\\ &\left\{\lim_{t\to+\infty} X_t \textit{ exists and finished}\right\}\cap\\ &\left\{\lim_{t\to+\infty} U_t<\infty\right\} a.s.,\end{aligned}$$

where, $C \subset D$ *a.s. means* $\mathbb{P}(C\cap D^c) = 0$. *In particular, if* $\lim_{t\to+\infty} A_t < \infty$ *a.s., then,* $\forall\omega \in \Omega$, $\lim_{t\to+\infty} X_t(\omega)$ *exists and finished and* $\lim_{t\to+\infty} U_t < \infty$.

Proof of theorem 3.2

Step 1 Applying Itô's formula to the second equation of system (11) and then taking an integration from 0 to t we get

$$\begin{aligned}&\ln(E_h(t)) &(25)\\ =&\int_0^t \beta_1 i_m(\xi)s_h(\xi)e_h^{-1}(\xi)d\xi &(26)\\ &-\int_0^t(\lambda+\mu_h+\frac{1}{2}\eta_2^2)d\xi &(27)\\ &-\int_0^t\frac{1}{2}\eta_1^2 i_m^2(\xi)s_h^2(\xi)e_h^{-2}(\xi)d\xi &\\ &+\ln(e_h(0))+\mathcal{M}_1(t)-\eta_2B_2(t) &(28)\end{aligned}$$

where

$$\mathcal{M}_1(t)=\int_0^t \eta_1 i_m(\xi)s_h(\xi)e_h^{-1}(\xi)dB_1(\xi) \tag{29}$$

is a continuous martingale (see [17]), whose quadratic variation is given by

$$\left\langle\mathcal{M}_1,\mathcal{M}_1\right\rangle(t)=\int_0^t \eta_1^2 i_m^2(\xi)s_h^2(\xi)e_h^{-2}(\xi)d(\xi).$$

For every integer $p \ge 1$, using the exponential martingale inequality *i.e* lemma 3.1, we see that

$$\mathcal{P}\Big\{\sup_{0\le t\le p}\left[\mathcal{M}_1-\frac{1}{4}\left\langle\mathcal{M}_1,\mathcal{M}_1\right\rangle(t)\right] > 4\ln p\Big\}\le\frac{1}{p^2}.$$

An application of the Borel-Cantelli lemma (see [17]) then yields that for almost all $\omega \in \Omega$ there exist a random integer $p_0 = p(\omega) \geq 1$ such that

$$\sup_{0\leq t\leq p}\left[\mathcal{M}_1 - \frac{1}{4}\left\langle \mathcal{M}_1, \mathcal{M}_1\right\rangle (t)\right] \leq 4\ln p, \quad \text{if } p \geq p_0 .$$

That is,

$$\mathcal{M}_1(t) \leq \frac{1}{4}\left\langle \mathcal{M}_1, \mathcal{M}_1\right\rangle (t) + 4\ln p, \quad (30)$$
$$\text{for all } 0 \leq t \leq p, \quad (31)$$
$$p \geq p_0 \text{ almost surly.} \quad (32)$$

Substituting (32) into (28) deduces that for all $0 \leq t \leq p$, $p \geq p_0$,

$$\begin{aligned}\ln\frac{e_h(t)}{e_h(0)} \leq & \int_0^t \beta_1 i_m(\xi)s_h(\xi)e_h^{-1}(\xi)d\xi \\ & -\int_0^t(\lambda+\mu_h+\frac{1}{2}\eta_2^2)d\xi \\ & -\frac{1}{4}\int_0^t \eta_1^2 i_m^2(\xi)s_h^2(\xi)e_h^{-2}(\xi)d\xi \\ & +4\ln p + \mathcal{M}_1(t) \\ & -\eta_2 B_2(t) \ .a.s.\end{aligned}$$

Set $u_1(\xi) = i_m(\xi)s_h(\xi)e_h^{-1}(\xi)$, then

$$\begin{aligned} & -\frac{1}{4}\eta_1^2 u_1^2(\xi) + \beta_1 u_1(\xi) \\ = & -\frac{1}{2}\eta_1^2\left[\left(u_1(\xi) - 2\frac{\beta_1}{\eta_1^2}\right)^2 - 4\left(\frac{\beta_1}{\eta_1^2}\right)^2\right] \\ \leq & \left(\frac{\beta_1}{\eta_1}\right)^2.\end{aligned}$$

It follows that

$$\begin{aligned}\ln\frac{E_h(t)}{e_h(0)} \leq & \left[\frac{\beta_1^2}{\eta_1^2} - \frac{1}{2}\left(\eta_2^2 + 2\lambda + 2\mu_h\right)\right]t \\ & -\eta_2 B_2(t) + 4\ln p \ .a.s.,\end{aligned}$$

for all $0 \leq t \leq p$, $p \geq P_0$ almost surly. Therefore, for almost all $\omega \in \Omega$, if $p \geq p_0$, $p-1 \leq t \leq p$,

$$\begin{aligned}\frac{1}{t}\ln E_h(t) \leq & \frac{1}{p-1}\left[\frac{\beta_1^2}{\eta_1^2} - \frac{1}{2}\left(\eta_2^2 + 2\lambda + 2\mu_h\right)\right]p \\ & -\eta_2\frac{1}{t}B_2(t) + 4\frac{\ln p}{p-1} + \frac{1}{t}\ln e_h(0) \\ & .a.s.\end{aligned} \quad (33)$$

Letting $t \longrightarrow \infty$, $p \longrightarrow \infty$ and using the well-known law, the strong law of large numbers (see [17]) to the Brownian motion we derive

$$\limsup_{t\to\infty}\frac{B_2(t)}{t} = 0 \ a.s. \quad (34)$$

Letting $p \to \infty$ then

$$\limsup_{p\to\infty}\frac{\ln p}{p-1} = 0 \ .a.s. \quad (35)$$

Taking the superior limit on both sides of (33) combining with (34) and (35) we get

$$\begin{aligned}\limsup_{t\to\infty}\frac{\ln(E_h(t))}{t} \leq & \frac{\beta_1^2}{\eta_1^2} - \\ & -\frac{1}{2}\left(\eta_2^2 + 2\lambda + 2\mu_h\right) \\ = & \frac{1}{2}\left(\eta_2^2 + 2\lambda + 2\mu_h\right) \times \\ & (\mathcal{R}_0^e - 1) \ a.s.\end{aligned}$$

Step 2 In a similar way, we apply the Itô's formula to the fourth equation of system (11) and then take an integration from 0 to t, which gives

$$\begin{aligned}\ln i_m(t) = & \ln i_m(0) + \mu_m t \\ & + \int_0^t \beta_3 i_h(\xi)s_m(\xi)i_m^{-1}(\xi)d\xi \\ & -\int_0^t \frac{1}{2}\eta_3^2 i_h^2(\xi)s_m^2(\xi)i_m^{-2}(\xi)d\xi \\ & +\int_0^t \beta_2 e_h(\xi)s_m(\xi)i_m^{-1}(\xi)d\xi \\ & -\frac{1}{2}\int_0^t \eta_4^2 e_h^2(\xi)s_m^2(\xi)i_m^{-2}(\xi)d\xi \\ & +\mathcal{M}_3(t) + \mathcal{M}_4(t),\end{aligned}$$

where

$$\mathcal{M}_3(t) = \int_0^t \eta_3 i_h(\xi)s_m(\xi)i_m^{-1}dB_3(\xi), \quad (36)$$
$$\mathcal{M}_4(t) = \int_0^t \eta_4 e_h s_m(\xi)i_m^{-1}(\xi)dB_4(\xi), \quad (37)$$

where, $\mathcal{M}_3(t)$ and $\mathcal{M}_4(t)$ are a continuous local martingales.

Set

$$\begin{aligned} y_3(t) &= \int_0^t \beta_3 i_h(\xi) s_m(\xi) i_m^{-1}(\xi) d\xi \\ &\quad -\frac{1}{2}\int_0^t \eta_3^2 i_h^2(\xi) s_m^2(\xi) i_m^{-2}(\xi) d\xi \\ &\quad -\mu_m d\xi + \mathcal{M}_3(t), \\ y_4(t) &= \int_0^t \beta_2 e_h(\xi) s_m(\xi) i_m^{-1}(\xi) d\xi \\ &\quad -\frac{1}{2}\int_0^t \eta_4^2 e_h^2(\xi) s_m^2(\xi) i_m^{-2}(\xi)\Big) d\xi \\ &\quad +\mathcal{M}_4(t) + \ln i_m(0)., \end{aligned}$$

Using the same method developed in step 1, we deduce that

$$\limsup_{t\to\infty} \frac{y_3(t)}{t} \leq \left(\frac{\beta_3}{\eta_3}\right)^2 - \mu_m \ a.s. \quad (38)$$

$$\limsup_{t\to\infty} \frac{y_4(t)}{t} \leq \left(\frac{\beta_4}{\eta_4}\right)^2 \ a.s. \quad (39)$$

Consequently,

$$\begin{aligned} &\limsup_{t\to\infty} \frac{\ln(i_m(t))}{t} \\ &\leq \Big[\left(\frac{\beta_3}{\eta_3}\right)^2 + \left(\frac{\beta_4}{\eta_4}\right)^2 - \mu_m\Big] \\ &= \mu_m\left(\mathcal{R}_0^m - 1\right) \ a.s. \end{aligned}$$

Step 3 Using the same calculation technique in step 1, we end up with

$$\begin{aligned} \limsup_{t\to\infty} \frac{i_h(t)}{t} &\leq \left(\frac{\lambda}{\eta_2}\right)^2 - \mu_h \ a.s. \quad (40) \\ &= \mu_h\left(\mathcal{R}_0^i - 1\right) \ a.s. (41) \end{aligned}$$

Hence if

$$\mathcal{R}_0^{Ex} < 1, \quad (42)$$

is true, the results of the steps (1-3) lead to the conclude that $e_h(t)$, $i_h(t)$ and $i_m(t)$ will tend to zero exponentially with probability one.

Step 4 In this step we need to show that 3.2, we just need to show that

$$\lim_{t\to\infty}(1 - s_h(t)) = 0. \ a.s. \quad (43)$$

By integrating the two sides of the first equation of system (11), we get

$$\begin{aligned} 1 - s_h(t) &= 1 - s_h(0) + \int_0^t \beta_1 s_h(\xi) i_m(\xi) d\xi \\ &\quad - \int_0^t \mu_H (1 - s_h(\xi)) d\xi \\ &\quad + \int_0^t \eta_1 s_h(\xi) i_m(\xi) dB_1(\xi). \end{aligned}$$

Since $s_h(t) < 1$, then we get

$$\lim_{t\to+\infty} \int_0^t \beta_1 s_h(\xi) i_m(\xi) d\xi < \lim_{t\to+\infty} \int_0^t \beta_1 i_m(\xi) d\xi.$$

Moreover, since $i_m(t)$ almost surely converges exponentially to 0, then there exists $c_1, c_2 > 0$ such that

$$i_m(\xi) < c_1 \exp(-c_2\xi) \ \forall \xi \geq 0.$$

It follows that,

$$\lim_{t\to+\infty} \int_0^t i_m(\xi) d\xi < \int_0^{+\infty} c_1 \exp(-c_2\xi) d\xi. (44)$$

Thus,

$$\begin{aligned} &\lim_{t\to+\infty} \int_0^t \beta_1 s_h(\xi) i_m(\xi) d\xi < \\ &\beta_1 \int_0^{+\infty} c_1 \exp(-c_2\xi) d\xi < \infty. \end{aligned}$$

Using the results of the lemma 3.2, we come to the conclusion

$$\begin{aligned} \lim_{t\to+\infty} (1 - s_h(t)) &< \infty \ a.s., \\ \lim_{t\to+\infty} \int_0^t \mu_h (1 - s_h(\xi)) d\xi &< \infty \ a.s. \\ \text{i.e } \int_0^\infty (1 - s_h(s)) d\xi &< \infty \ a.s. (45) \end{aligned}$$

Assume that $(s_h(t))_{t\geq 0}$ does not converge to one. Then there exists $C \subset \Omega$ with $\mathbb{P}(C) > 0$ such as, $\forall \omega \in C$,

$$\liminf_{t\to\infty} (1 - s_h(t,\omega)) = \varrho(w) > 0.$$

Thus there exists $T = T_\omega > 0$ such that $(1 - s_h(t,\omega)) = \frac{1}{2}\varrho(w) > 0$, $\forall t \geq T$. Hence,

$$\begin{aligned} \int_0^\infty (1 - s_h(\xi,\omega)) d\xi &= \int_0^T (1 - s_h(s,\omega)) d\xi \\ &\quad + \int_T^\infty (1 - s_h(\xi,\omega)) d\xi \\ &> \int_T^\infty (1 - s_h(\xi,\omega)) d\xi \\ &= \infty. \end{aligned}$$

This implies that $C \subset D$ where, $D := \left\{\omega \in \Omega : \int_0^\infty (1 - s_h(\xi,\omega))d\xi = \infty\right\}$. Yet inequality (45), $\mathbb{P}(D) = 0$, leads to a contradiction. So, $\lim_{t\to\infty}(1 - S_h(t)) = 0$ *a.s.* Using similar reasoning, we show that $\lim_{t\to\infty}(1-s_m(t)) = 0$ *a.s.* This completes the proof.

3.3 Persistence in mean of filiriasis

Before establishing the persistence results, we will state a lemma that will be used in the proofs.

Lemma 3.3 *Let* $(s_h(.), e_h(.), i_h(.), s_m(.), i_m(.))$ *a solution of* (11) *with initial conditions* $(s_h(0), e_h(0), i_h(0), s_m(0), i_m(0)) \in (0;1)^5$. *Then*

$$\lim_{t\to+\infty} \frac{s_h(t)}{t} = 0, \quad (46)$$
$$\lim_{t\to+\infty} \frac{e_h(t)}{t} = 0,$$
$$\lim_{t\to+\infty} \frac{i_h(t)}{t} = 0, \quad (47)$$
$$\lim_{t\to+\infty} \frac{s_m(t)}{t} = 0,$$
$$\lim_{t\to+\infty} \frac{i_m(t)}{t} = 0. \ a.s. \quad (48)$$

Proof. Our approach is inspired by the works of Yanan Zhao and Daqing Jiang (see [23]) and Yanli Zhou and Weiguo Zhang (see [25]).
Let $u(t) = s_h(t)+e_h(t)+i_h(t)+s_m(t)+i_m(t)$. Define $V(u(t)) = (1 + u(t))^\theta$ where, θ is a positive constant.
Applying Itô's formula to V, we get

$$dV(u(t)) = \theta(1 + u(t))^{\theta-1}du + \frac{1}{2}\theta(\theta - 1)(1 + u(t))^{\theta-2} \times (du(t))^2.$$

We have

$$du(t) = (\mu_h + \mu_m - \mu_h n_h(t) - \mu_m n_m(t))\, dt$$

where,

$$n_h(t) = s_h(t) + e_h(t) + i_h(t)$$
$$\text{and } n_m(t) = s_m(t) + i_m(t).$$

We get
$$(du(t))^2 = 0.$$
So,
$dV(u(t)) = \theta(1 + u(t))^{\theta-1}du$
$= \mathcal{L}V(u(t))\, dt.$
Where,
$\mathcal{L}V(u(t)) \leq \theta(1 + u(t))^{\theta-1} \times [(\mu_h + \mu_m) - \mu_h n_h(t)] - \mu_m n_m(t).$

Set $\mu_1 = \max(\mu_h, \mu_m)$, $\mu_2 = \min(\mu_h, \mu_m)$.
The following mark-up is obtained

$$\begin{aligned}\mathcal{L}V(u(t)) &\leq \theta(1 + u(t))^{\theta-1} \times [2\mu_1 - \mu_2 n_h(t) - \mu_m n_m(t)]\\ &\leq \theta(1 + u(t))^{\theta-1} \times [2\mu_1 - \mu_2 u(t)]\\ &\leq \theta(1 + u(t))^{\theta-2} \times [(1 + u(t))(2\mu_1 - \mu_2 u(t))]\\ &\leq \theta(1 + u(t))^{\theta-2} \times [2\mu_1 + (2\mu_1 - \mu_2)u(t)] - \mu_2 u^2(t).\end{aligned}$$

It follows that

$$dV(u(t)) \leq \theta(1 + u(t))^{\theta-2} \times \quad (49)$$
$$[2\mu_1 + (2\mu_1 - \mu_2)u(t)]\, dt - \mu_2 u^2(t)dt. \quad (50)$$

For $p > 0$, we get

$$\begin{aligned}& d\left[e^{pt}V(u(t))\right] \quad (51)\\ =\ & \mathcal{L}\left[e^{pt}V(u(t))\right] dt\\ =\ & pe^{pt}V(u(t))\, dt + e^{pt}dV(u(t))\, dt\\ \leq\ & pe^{pt}(1 + u(t))^\theta \quad (52)\\ & +\theta e^{pt}(1 + u(t))^{\theta-2} \times [2\mu_1 + (2\mu_1 - \mu_2)u(t)]\, dt - \mu_2 u^2(t)dt\\ \leq\ & \theta e^{pt}(1 + u(t))^{\theta-2} \times \left[\frac{p}{\theta}(1 + u(t))^2 - \mu_2 u^2(t)\right] dt + [(2\mu_1 - \mu_2)u(t) + 2\mu_1]\, dt\\ \leq\ & \theta e^{pt}(1 + u(t))^{\theta-2} \times \left[-\left(\mu_2 - \frac{p}{\theta}\right) u^2(t)\right] dt + (2\mu_1 - \mu_2 + 2\frac{p}{\theta})u(t)dt + \left(2\mu_1 + \frac{p}{\theta}\right) dt\\ \leq\ & \theta e^{pt}H dt, \quad (53)\end{aligned}$$

where,

$$H := \sup_{t\in\mathbb{R}_+}\Big\{(1+u(t))^{\theta-2}\Big[-\Big(\mu_2-\frac{p}{\theta}\Big)u^2(t) +(2\mu_1-\mu_2+2\frac{p}{\theta})u(t)+\Big(2\mu_1+\frac{p}{\theta}\Big)\Big]\Big\}.$$

Since $(s_h(.), e_h(.), i_h(.), s_m(.), i_m(.)) \in (0,1)^5$, then $u(.) \in (0, 25)$.
So, $0 < H < \infty$. Passing to the integral between 0 and t in (53), we get

$$\int_0^t d\left[e^{p\xi}V(u(\xi))\,d\xi\right\} \leq \int_0^t \theta e^{p\xi}H d\xi, \quad (54)$$

$$e^{pt}V(u(t)) \leq V(u(0)) + \frac{\theta H e^{pt}}{p} - \frac{\theta H}{p}.$$

It can be deduced that

$$Ee^{pt}V(u(t)) \leq V(u(0)) + \frac{\theta H e^{pt}}{p} - \frac{\theta H}{p}. \quad (55)$$

That is to say,

$$E\left[(1+u(t))^\theta\right] \leq \frac{(1+u(0))^\theta}{e^{pt}} + \frac{\theta H}{P} \leq (1+u(0))^\theta + \theta H. \quad (56)$$

Set $C = (1+u(0))^\theta + \theta H$.
Then,

$$E\left[(1+u(t))^\theta\right] \leq C. \quad (57)$$

$\forall\ \delta > 0$ sufficiently small, $p = 1, 2, 3, \ldots$, by integrating (50) between $p\delta$ and t, we get

$$(1+u(t))^\theta \leq (1+u(p\delta))^\theta + \int_{p\delta}^t \theta(1+u(\xi))^{\theta-2} \times \left[2\mu_1+(2\mu_1-\mu_2)u(\xi)-\mu_2u^2(\xi)\right]d\xi.$$

Taking the upper bound for t between $p\delta$ and $(p+1)\delta$, we get

$$\sup_{p\delta\leq t\leq(p+1)\delta}(1+u(t))^\theta \leq (1+u(p\delta))^\theta + \sup_{p\delta\leq t\leq(p+1)\delta}\Big|\int_{p\delta}^t \theta(1+u(\xi))^{\theta-2} \times \left[2\mu_1+(2\mu_1-\mu_2)u(\xi)-\mu_2u^2(\xi)\right]d\xi\Big|.$$

Taking the mathematical expectation of both sides of the latter inequality we get

$$E\left[\sup_{p\delta\leq t\leq(p+1)\delta}(1+u(t))^\theta\right] \leq E\left[(1+u(p\delta))^\theta\right] + J,$$

where,

$$\begin{aligned} J &= E\Big[\sup_{p\delta\leq t\leq(p+1)\delta}\Big|\int_{p\delta}^t \theta(1+u(\xi))^{\theta-2} \left[2\mu_1+(2\mu_1-\mu_2)u(\xi)-\mu_2u^2(\xi)\right]d\xi\Big|\Big] \\ &= E\Big[\sup_{p\delta\leq t\leq(p+1)\delta}\Big|\int_{p\delta}^t \theta(1+u(\xi))^{\theta-2} (1+u(\xi))(2\mu_1-\mu_2u(\xi))\,d\xi\Big|\Big] \\ &= E\Big[\sup_{p\delta\leq t\leq(p+1)\delta}\Big|\int_{p\delta}^t \theta(1+u(\xi))^{\theta-1} (2\mu_1-\mu_2u(\xi))\,d\xi\Big|\Big] \\ &= E\Big[\sup_{p\delta\leq t\leq(p+1)\delta}\Big|\int_{p\delta}^t \theta(1+u(\xi))^{\theta} \times \frac{2\mu_1-\mu_2u(\xi)}{(1+u(\xi))}d\xi\Big|\Big]. \end{aligned}$$

$$\text{Set : } l = \theta \sup_{p\delta\leq t\leq(p+1)\delta}\left|\frac{2\mu_1-\mu_2u(\xi)}{(1+u(\xi))}\right|. \quad (58)$$

It follows that

$$\begin{aligned} J &\leq lE\left[\sup_{p\delta\leq t\leq(p+1)\delta}\left|\int_{p\delta}^t(1+u(\xi))^\theta d\xi\right|\right] \\ &\leq lE\left[\int_{p\delta}^{(p+1)\delta}(1+u(\xi))^\theta d\xi\right] \\ &\leq lE\left[\delta\sup_{p\delta\leq\xi\leq(p+1)\delta}(1+u(\xi))^\theta\right] \\ &\leq l\delta E\left[\sup_{p\delta\leq\xi\leq(p+1)\delta}(1+u(\xi))^\theta\right] \\ &\leq l\delta E\left[\sup_{p\delta\leq t\leq(p+1)\delta}(1+u(t))^\theta\right]. \end{aligned}$$

As a result

$$\begin{aligned} &E\left[\sup_{p\delta\leq t\leq(p+1)\delta}(1+u(t))^\theta\right] \\ \leq\ &E\left[(1+u(p\delta))^\theta\right] + l\delta E\left[\sup_{p\delta\leq t\leq(p+1)\delta}(1+u(t))^\theta\right]. \end{aligned}$$

Choose $\delta > 0$ such as $l\delta \leq \frac{1}{2}$, then

$$E\left[\sup_{p\delta\leq t\leq(p+1)\delta}(1+u(t))^\theta\right] \leq 2E\left[(1+u(p\delta))^\theta\right].$$

By using (57), we get

$$E\left[\sup_{p\delta\le t\le (p+1)\delta}(1+u(t))^{\theta}\right]\le 2C. \qquad (59)$$

Let ϵ_u an arbitrarily chosen positive constant. Applying Markov's inequality, we get

$$\begin{aligned} & P\left\{\sup_{p\delta\le t\le (p+1)\delta}(1+u(t))^{\theta}>(p\delta)^{1+\epsilon_u}\right\} \\ \le\ & \frac{E\left[\sup_{p\delta\le t\le (p+1)\delta}(1+u(t))^{\theta}\right]}{(p\delta)^{1+\epsilon_u}} \qquad (60) \\ \le\ & \frac{2C}{(p\delta)^{1+\epsilon_u}}. \qquad (61) \end{aligned}$$

Let $U_p = \{\sup_{p\delta\le t\le (p+1)\delta}(1+u(t))^{\theta} > (p\delta)^{1+\epsilon_u}\}$ then $\sum_{p=1}^{\infty}P(U_p)<\sum_{p=1}^{\infty}\frac{2C}{(p\delta)^{1+\epsilon_u}}$.

Since $1+\epsilon_u>1$ then $\sum_{p=1}^{\infty}\frac{2C}{(p\delta)^{1+\epsilon_u}}<\infty$ the Borel-Cantelli lemma (see [17]) yields that for almost all $\omega\in\Omega$

$$\sup_{p\delta\le t\le (p+1)\delta}(1+u(t))^{\theta}\le (p\delta)^{1+\epsilon_u}, \quad p=1,2,3,... \qquad (62)$$

Since this inequality holds for all p, then there exists a positive inter $p_0=p_0(\omega)$ for almost all $\omega\in\Omega$ such that (62) remains true, $\forall p\ge p_0$. Therefore, for almost all $\omega\in\Omega$, if $p\ge p_0$ and $p\delta\le t\le (p+1)\delta$,

$$\begin{aligned}\frac{\ln(1+u(t))^{\theta}}{\ln t} &\le \frac{(1+\epsilon_u)\ln(p\delta)}{\ln(p\delta)} \\ &= 1+\epsilon_u.\end{aligned}$$

So,

$$\limsup_{t\to\infty}\frac{\ln(1+u(t))^{\theta}}{\ln t}\le 1+\epsilon_u, \ a.s. \qquad (63)$$

Let's make $\epsilon_u\longrightarrow 0$, we get

$$\limsup_{t\to\infty}\frac{\ln(1+u(t))^{\theta}}{\ln t}\le 1, \ a.s.$$

For $\theta>1$, we get

$$\begin{aligned}\limsup_{t\to\infty}\frac{\ln(u(t))}{\ln t} &\le \limsup_{t\to\infty}\frac{\ln(1+u(t))}{\ln t} \\ &\le \frac{1}{\theta}, \ a.s.\end{aligned}$$

That is to say, for $0<\gamma<1-\frac{1}{\theta}$, there exists a constant $T=T(\omega)$ such as, $\forall t\ge T$

$$\ln(1+u(t))\le\left(\frac{1}{\theta}+\gamma\right)\ln t. \qquad (64)$$

That is to say, for $0<\gamma<1-\frac{1}{\theta}$, there exists a constant $T=T(\omega)$ and a set Ω_γ such as $P(\Omega_\gamma)\ge 1-\gamma$ and, $\forall t\ge T$, $\omega\in\Omega_\gamma$,

$$\ln(u(t))\le\left(\frac{1}{\theta}+\gamma\right)\ln t. \qquad (65)$$

As a result

$$\limsup_{t\to\infty}\frac{u(t)}{t}\le\limsup_{t\to\infty}\frac{t^{\frac{1}{\theta}+\gamma}}{t}=0. \qquad (66)$$

This leads to

$$\lim_{t\to\infty}\frac{u(t)}{t} = \lim_{t\to\infty}\frac{s_h(t)+e_h(t)+i_h(t)+s_m(t)+i_m(t)}{t}=0. \quad a.s.,$$

thanks to the positivity of s_h, e_h, i_h, s_m and i_m. So, we get

$$\lim_{t\to\infty}\frac{s_h(t)}{t}=\lim_{t\to\infty}\frac{e_h(t)}{t}=\lim_{t\to\infty}\frac{i_h(t)}{t}= \lim_{t\to\infty}\frac{s_m(t)}{t}=\lim_{t\to\infty}\frac{i_m(t)}{t}=0 \ a.s.$$

This completes the proof.

Lemma 3.4 *Let* $(s_h(.),e_h(.),i_h(.),s_m(.),i_m(.))$ *a solution of system* (11) *with initial conditions* $(s_h(0),e_h(0),i_h(0),s_m(0),i_m(0)) \in (0;1)^5$. *Then*

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}\int_0^t i_m(\xi)s_h(\xi)e_h^{-1}(\xi)dB_1(\xi)=0 \ a.s., \\ &\lim_{t\to\infty}\frac{1}{t}\int_0^t e_h(\xi)i_h^{-1}(\xi)dB_(\xi)=0 \ .a.s.\end{aligned}$$

Proof. Let

$$\begin{aligned}\mathcal{M}_1(t) &= \int_0^t i_m(\xi)s_h(\xi)e_h^{-1}(\xi)dB_1(\xi), \\ \mathcal{M}_2(t) &= \int_0^t e_h(\xi)i_h^{-1}(\xi)dB_(\xi).\end{aligned}$$

As the maps i_h, s_h and i_m are continuous, then by using the Weierstrass theorem, we get

$$\sup_{0\le\xi\le t}\{i_m(\xi)s_h(\xi)e_h^{-1}(\xi)\} = C_1 < \infty.$$

Thus

$$\begin{aligned}\langle\mathcal{M}_1(t),\mathcal{M}_1(t)\rangle &< C_1 t.\ a.s. \text{ and}\\ \limsup_{t\to\infty}\frac{\langle\mathcal{M}_1(t),\mathcal{M}_1(t)\rangle}{t} &< C_1.\ a.s.\end{aligned}$$

By the strong law of large numbers for local martingales, we conclude that

$$\lim_{t\to\infty}\frac{\mathcal{M}_1(t)}{t} = 0.\ a.s. \tag{67}$$

In the same way we get

$$\lim_{t\to\infty}\frac{\mathcal{M}_2(t)}{t} = 0.\ a.s. \tag{68}$$

Hence the lemma has been established.

Lemma 3.5 *Let $f\in\mathcal{C}([0;\infty)\times\Omega,(0,\infty))$. If there are positive constants λ_0, λ and T such that*

$$\ln f(t) \ge \lambda t - \lambda_0\int_0^t f(\xi)d\xi + F(t), \tag{69}$$

$\forall t \ge T$ with $F \in \mathcal{C}([0;\infty)\times\Omega,\mathbb{R})$, $\lim_{t\to\infty}\frac{F(t)}{t} = \ell$ a.s. and $\lambda+\ell>0$. Then

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t f(\xi)d\xi \ge \frac{\lambda+\ell}{\lambda_0}\ a.s.$$

Proof

Our approach is inspired by the work of Zhaoa, Daqing Jiang and Donal O'Reganc (see [24]) and Liu Huaping and Ma Zhien (see[13]) .

Note that $\lim_{t\to\infty}\frac{F(t)}{t} = \ell$ a.s. then for arbitrary $0<\epsilon<\lambda+\ell$ there exists a $T_0 = T_0(\omega)>0$ and a set Ω_r such that $P(\Omega_r)>0$ and $\left|\frac{F(t)}{t}-\ell\right| \le \epsilon$ for all $t\ge T_0$, $\omega\in\Omega_\epsilon$.

Let $\overline{T} = T\vee T_0$ and $\psi(t) = \int_0^t f(\zeta)d\zeta$ for $t\ge\overline{T}$, $\omega\in\Omega_r$.

Since $f\in C([0,\infty)\times\Omega,(0,\infty))$, then ψ is differentiable on $[\overline{T},\infty)$ a.s. and

$$d\psi(t) = f(t) > 0 \text{ for } t\ge\overline{T},\omega\in\Omega_r.$$

Substituting $\frac{d\psi(t)}{dt}$ and $\psi(t)$ into (69), we have

$$\begin{aligned}\ln\left(\frac{d\psi(t)}{dt}\right) &\ge \lambda t - \lambda_0\psi(t) + F(t)\\ &\ge (\lambda-\epsilon+\ell)t - \lambda_0\psi(t),\\ &\quad \text{for } t\ge\overline{T},\omega\in\Omega_r.\end{aligned}$$

So

$$\begin{aligned}\exp(\lambda_0\psi(t))\frac{d\psi(t)}{dt} &\ge \exp(\lambda-\epsilon+\ell)t,\\ &\quad\text{for } t\ge\overline{T},\omega\in\Omega_r.\end{aligned}$$

Integrating this inequality from $\overline{T}$ to t results in

$$\lambda_0^{-1}\Big[\exp(\lambda_0\psi(t)) - \exp(\lambda_0\psi(\overline{T}))\Big] \ge \Big[\exp((\lambda+\ell-\epsilon)t) - \exp((\lambda+\ell-\epsilon)\overline{T})\Big].$$

This inequality can be rewritten into

$$\begin{aligned}&\exp(\lambda_0\psi(t)) \ge\\ &\lambda_0(\lambda+\ell-\epsilon)^{-1}\times\\ &\Big[\exp((\lambda+\ell-\epsilon)t) - \exp((\lambda+\ell-\epsilon)\overline{T})\Big]\\ &+\exp(\lambda_0\psi(\overline{T}).\end{aligned}$$

Taking the logarithm of both sides yields

$$\psi(t) \ge \lambda_0^{-1}\ln\Big[\lambda_0(\lambda+\ell-\epsilon)^{-1}\times \exp((\lambda+\ell-\epsilon)t) + \lambda_{\overline{T}}\Big],$$

where,

$$\begin{aligned}\lambda_{\overline{T}} &= \exp(\lambda_0\psi(\overline{T}))\\ &\quad -\lambda_0(\lambda+\ell-\epsilon)^{-1}\exp((\lambda+\ell-\epsilon)\overline{T})\end{aligned}$$

or

$$\int_0^t f(\zeta)d\zeta \ge \lambda_0^{-1}\ln\Big[\lambda_0(\lambda+\ell-\epsilon)^{-1}\times \exp((\lambda+\ell-\epsilon)t) + \lambda_{\overline{T}}\Big],\ \text{for } t\ge\overline{T},\omega\in\Omega_r.$$

Dividing both sides by $t\ge\overline{T}>0$ gives

$$t^{-1}\int_0^t f(\zeta)d\zeta \ge \lambda_0^{-1}t^{-1}\ln\Big[\lambda_0(\lambda+\ell-\epsilon)^{-1}\times \exp((\lambda+\ell-\epsilon)t) + \lambda_{\overline{T}}\Big],\ \text{for } t\ge\overline{T},\omega\in\Omega_r.$$

Taking the limit inferior of both sides and applying L'Hospital's rule on the right-hand side of this inequality, we obtain

$$\liminf_{t\to\infty} t^{-1}\int_0^t f(\zeta)d\zeta \geq \frac{\lambda+\ell-\epsilon}{\lambda_0} \text{ for } \omega\in\Omega_r.$$

Letting $\epsilon\to 0$ yields

$$\liminf_{t\to\infty} t^{-1}\int_0^t f(\zeta)d\zeta \geq \frac{\lambda+\ell}{\lambda_0} \text{ for } a.s. \quad (70)$$

This finishes the proof of the lemma.

We now turn to the study of the persistence in mean of system (11). To this end, we present a definition of persistence in mean that can be found in [7, 24]. For future needs, define the following threshold parameter (this threshold parameter is derived from the proof of the theorem 3.3 please see (80)),

$$\mathcal{R}_0^s = \beta_1^{-1}\left(\lambda+\mu_h+\frac{1}{2}(\eta_2^2+\epsilon\eta_1^2)\right)$$

and formulate the following hypotheses

- $(\mathcal{H}_1)$ there exist $\epsilon>0$ such as $\mid s_h(t)i_m(t)e_h^{-1}\mid\leq\epsilon$,
- $(\mathcal{H}_2)$ $i_m(\xi)e_h^{-1}(\xi)\geq 1$,
- $(\mathcal{H}_3)$ $\mathcal{R}_0^s<1$.

Theorem 3.3 *Let* $(s_h(.),e_h(.),i_h(.),s_m(.),i_m(.))$ *a solution of system* (11) *with the initial conditions* $(s_h(0),e_h(0),i_h(0),s_m(0),i_m(0))\in(0;1)^5$. *If the assumptions* $(\mathcal{H}_1)-(\mathcal{H}_3)$ *are verified then*

$$\begin{aligned}\liminf_{t\longrightarrow\infty}\langle e_h(t)\rangle &> 0\ a.s.,\\ \liminf_{t\longrightarrow\infty}\langle i_h(t)\rangle &> 0\ a.s.\end{aligned}$$

Proof

Applying the integral between 0 to t the two sides of the three first equations of system (11) we get

$$\begin{aligned}\frac{s_h(t)-s_h(0)}{t} &= \mu_h-\beta_1\langle i_m(t)s_h(t)\rangle\\ &\quad -\frac{\eta_1}{t}\int_0^t i_m(\xi)s_h(\xi)dB_1(\xi)\\ &\quad -\mu_h\langle s_h(t)\rangle,\\ \frac{e_h(t)-e_h(0)}{t} &= \beta_1\langle i_m(t)s_h(t)\rangle\\ &\quad -(\lambda+\mu_h)\langle e_h(t)\rangle \quad (71)\\ &\quad +\frac{\eta_1}{t}\int_0^t i_m(\xi)s_h(\xi)dB_1(\xi)\\ &\quad -\frac{\eta_2}{t}\int_0^t e_h(\xi)dB_2(\xi),\\ \frac{i_h(t)-i_h(0)}{t} &= \lambda\langle e_h(t)\rangle-\mu_h\langle i_h(t)\rangle \quad (72)\\ &\quad +\frac{\eta_2}{t}\int_0^t e_h(\xi)dB_2(\xi), \quad (73)\end{aligned}$$

Thus

$$\begin{aligned}&\frac{s_h(t)-s_h(0)}{t}+\frac{e_h(t)-e_h(0)}{t}+\\ &+\frac{i_h(t)-i_h(0)}{t}=\mu_h-\mu_h\langle s_h(t)\rangle\\ &-\mu_h\langle e_h(t)\rangle-\mu_h(\langle i_h(t)\rangle)\end{aligned}$$

Which yields

$$\langle s_h(t)\rangle = 1-\langle e_h(t)\rangle-\langle i_h(t)\rangle+\phi(t), \quad (74)$$

where

$$\begin{aligned}\phi(t) = -\frac{1}{\mu_h}\Big[&\frac{s_h(t)-s_h(0)}{t}+\\ &\frac{e_h(t)-e_h(0)}{t}+\frac{i_h(t)-i_h(0)}{t}\Big].\end{aligned}$$

Using lemma (3.3), we get

$$\lim_{t\longrightarrow\infty}\phi(t)=0\ a.s. \quad (75)$$

By applying the Itô's formula to system (11) leads to

$$\begin{aligned}d\ln(e_h(t)) &= \frac{1}{e_h(t)}de_h-\frac{1}{2}\frac{1}{e_h^2(t)}(de_h)^2\\ &= \beta_1 i_m(t)s_h(t)e_h^{-1}(t)dt\\ &\quad \left[-(\lambda+\mu_h)-\frac{1}{2}\eta_2^2\right]dt\\ &\quad -\frac{1}{2}\eta_1^2 i_m^2(t)s_h^2(t)e_h^{-2}(t)dt\\ &\quad +\eta_1 i_m(t)s_h(t)e_h^{-1}(t)dB_1(t)\\ &\quad -\eta_2 dB_2(t).\end{aligned}$$

Passing to integral from 0 to t, it follow that

$$\begin{aligned}&t^{-1}\left(\ln e_h(t)-\ln e_h(0)\right)=\\&\beta_1\left\langle i_m(t)s_h(t)e_h^{-1}(t)\right\rangle\\&-\frac{1}{2}\eta_1^2\left\langle i_m^2(t)s_h^2(t)e_h^{-2}(t)\right\rangle-\frac{\eta_2}{t}B_2(t)\\&+\frac{\eta_1}{t}\int_0^t i_m(\xi)s_h(\xi)e_h^{-1}(\xi)dB_1(\xi)\\&-(\lambda+\mu_h+\frac{1}{2}\eta_2^2).\end{aligned}$$

If $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ are hold, then we obtain

$$\begin{aligned}&t^{-1}\left(\ln e_h(t)-\ln e_h(0)\right)\geq\\&\beta_1\left\langle s_h(t)\right\rangle-\frac{1}{2}\epsilon\eta_1^2\\&-\frac{\eta_2}{t}B_2(t)-(\lambda+\mu_h+\frac{1}{2}\eta_2^2)\\&+\frac{\eta_1}{t}\int_0^t i_m(\xi)s_h(\xi)e_h^{-1}(\xi)dB_1(\xi).\end{aligned}$$

Replacing $\langle s_h(t)\rangle$ inside , gives

$$\begin{aligned}&t^{-1}\left(\ln e_h(t)-\ln e_h(0)\right)\geq\\&\beta_1-\beta_1\left\langle e_h(t)\right\rangle-\beta_1\left\langle i_h(t)\right\rangle\\&-\frac{1}{2}\epsilon\eta_1^2-\frac{\eta_2}{t}B_2(t)\\&+\frac{\eta_1}{t}\int_0^t i_m(\xi)s_h(\xi)e_h^{-1}(\xi)dB_1(\xi)\\&-(\lambda+\mu_h+\frac{1}{2}\eta_2^2).\end{aligned}$$

From (73) we have

$$\langle I_h(t)\rangle=\frac{\lambda}{\mu_h}\left\langle E_h(t)\right\rangle+\frac{1}{\mu_h}\Big[\frac{1}{t}\int_0^t e_h(\xi)dB_2(\xi)-\frac{I_h(t)-i_h(0)}{t}\Big].$$

Using (74) we get

$$\begin{aligned}&t^{-1}\left(\ln e_h(t)-\ln e_h(0)\right)\\\geq\ &\beta_1-\left(\lambda+\mu_h+\frac{1}{2}\eta_2^2\right)\\&-\beta_1\left(1+\frac{\lambda}{\mu_h}\right)\left\langle e_h(t)\right\rangle-\beta_1\phi(t)\\&-\frac{\eta_2}{t}B_2(t)-\frac{\beta_1}{\mu_h}\left(\frac{1}{t}\int_0^t e_h(\xi)dB_2(\xi)\right)\\&-\frac{1}{2}\epsilon\eta_1^2+\frac{\eta_1}{t}\int_0^t i_m(\xi)s_h(\xi)e_h^{-1}(\xi)dB_1(\xi)\\&+\frac{\beta_1}{\mu_h}\left(\frac{I_h(t)-i_h(0)}{t}\right).\end{aligned}$$

It follow that

$$\begin{aligned}\ln e_h(t)\ \geq\ &\left[\beta_1-\left(\lambda+\mu_h+\frac{1}{2}\eta_2^2\right)\right]t\\&-\beta_1\left(1+\frac{\lambda}{\mu_h}\right)\int_0^t e_h(\xi)d\xi\\&+F(t).\end{aligned}$$

where,

$$\begin{aligned}F(t)\ =\ &\ln e_h(0)-\frac{1}{2}\epsilon\eta_1^2 t\\&+\beta_1 t\phi(t)-\eta_2 B_2(t);\\&+\eta_1\int_0^t i_m(\xi)s_h(\xi)e_h^{-1}(\xi)dB_1(\xi)\\&-\frac{\beta_1}{\mu_h}\Big[\int_0^t e_h(\xi)dB_2(\xi)\Big]\\&+\frac{\beta_1}{\mu_h}\left(I_h(t)-i_h(0)\right).\end{aligned}$$

Clearly

$$\lim_{t\to\infty}\frac{i_h(t)-i_h(0)}{t}=0\ a.s.\ \text{and}\qquad(76)$$

$$\lim_{t\to\infty}\frac{\ln(e_h(0)}{t}=0\ a.s.\qquad(77)$$

Addition, in view of the strong law of large numbers (see [16]) to the Brownien motion, we derive

$$\lim_{t\to\infty}\frac{B_2(t)}{t}=0\ a.s.\qquad(78)$$

Following the well-know law, the local martingales of large number theorem, we obtain

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t i_m(\xi)s_h(\xi)e_h^{-1}(\xi)dB_1(\xi)=0\ a.s.$$
$$\lim_{t\to\infty}\frac{1}{t}\int_0^t e_h(\xi)dB_2(\xi)=0\ a.s.$$

So,

$$\lim_{t\to\infty}\frac{F(t)}{t}=-\frac{1}{2}\epsilon\eta_1^2\ a.s.\qquad(79)$$

Using the result of the lemma 3.5 we get

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{t}\int_0^t e_h(\xi)d\xi\\\geq\ &\frac{\left[\beta_1-\left(\lambda+\mu_h+\frac{1}{2}(\eta_2^2+\epsilon\eta_1^2)\right)\right]}{\beta_1\left(1+\frac{\lambda}{\mu_h}\right)}\\=\ &\frac{\mu_h}{1+\lambda}\left[1-\beta_1^{-1}\left(\lambda+\mu_h+\frac{1}{2}(\eta_2^2+\epsilon\eta_1^2)\right)\right]\\=\ &\frac{\mu_h}{1+\lambda}\left(1-\mathcal{R}_0^s\right)\ a.s.\end{aligned}\qquad(80)$$

If $(\mathcal{H}_3)$ is hold, then we get

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t e_h(\xi)d\xi > 0 \ a.s. \qquad (81)$$

Next, from we can obtain

$$\begin{aligned}\liminf_{t\to\infty} \langle i_h(t)\rangle &= \frac{\lambda}{\mu_h}\liminf_{t\to\infty} \langle e_h(t)\rangle \ a.s.\\ &> 0 \ a.s.\end{aligned}$$

This completes the proof.

4 Numerical simulations

In the current section, we perform numerical simulations of the model (8). We use the MATLAB software and the technique developed in [11]. The values of the parameters taken to verify the extinction result in the deterministic model are recorded in table 4. By a simple calculation we verify that the basic reproduction number $\mathcal{R}_0 < 1$. According to the theorem 2.2 the Lymphatic filariasis will disappears. The theorem is therefore highlighted in Figure 2.

In the stochastic model, to verify the extinction result, we keep the same values of the indicated parameters in table 4. The values of the white noise intensities used are as follows

$$\begin{aligned}\eta_1 = 0.3,\ \eta_2 &= 0.125,\\ \eta_3 = 0.225,\ \eta_4 &= 0.225.\end{aligned}$$

(see [9] for the data).

By a simple calculation we obtain $\mathcal{R}_0^i = 0.01$, $\mathcal{R}_0^m = 0.0013$, $\mathcal{R}_0^e = 5.16.10^{-4}$ and $\mathcal{R}_0^{Ex} = 0.01$. According to the theorem 3.2 the disease will extinct. Figure 2 shows that the numerical simulations enhance this theorem.

Similarly, the parameter values chosen to the persistence are also recorded in table 4. For the choice of the values of the white noise intensities we rely on the data used in [9]. We suppose that

$$\begin{aligned}\eta_1 = 0.03,\ \eta_2 &= 0.025,\\ \eta_3 = 0.0225,\ \eta_4 &= 0.025.\end{aligned}$$

We can simply be shown by a calculation that $\mathcal{R}_0^s = 0.68 < 1$. Theorem 3.3 implies that model (11) is persistent in mean which is supported by the figure 4.

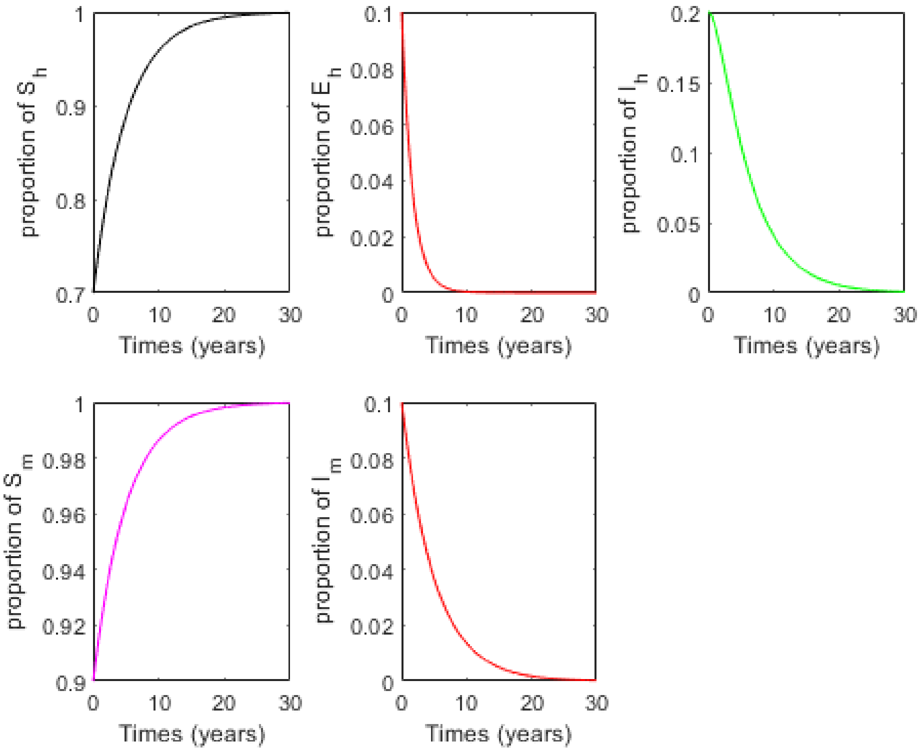

Figure 2: Plots showing the transmission dynamics of filariasis in the deterministic case when $\mathcal{R}_0 = 0.068$.

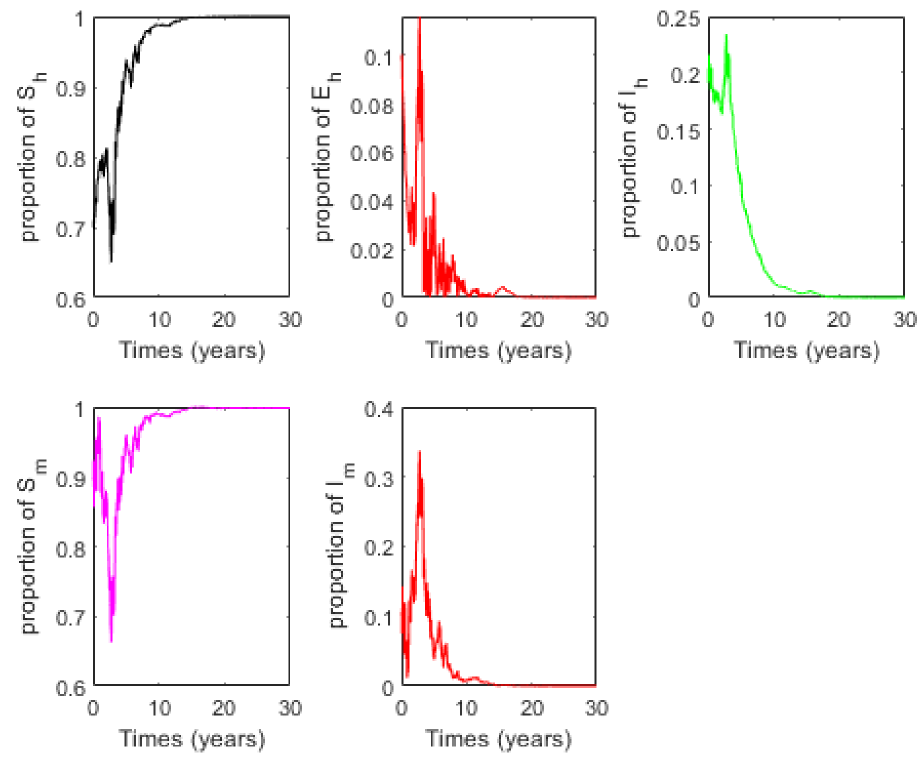

Figure 3: Graphs showing the transmission dynamics of filariasis for $\mathcal{R}_0^{Ex} = 0.01$.

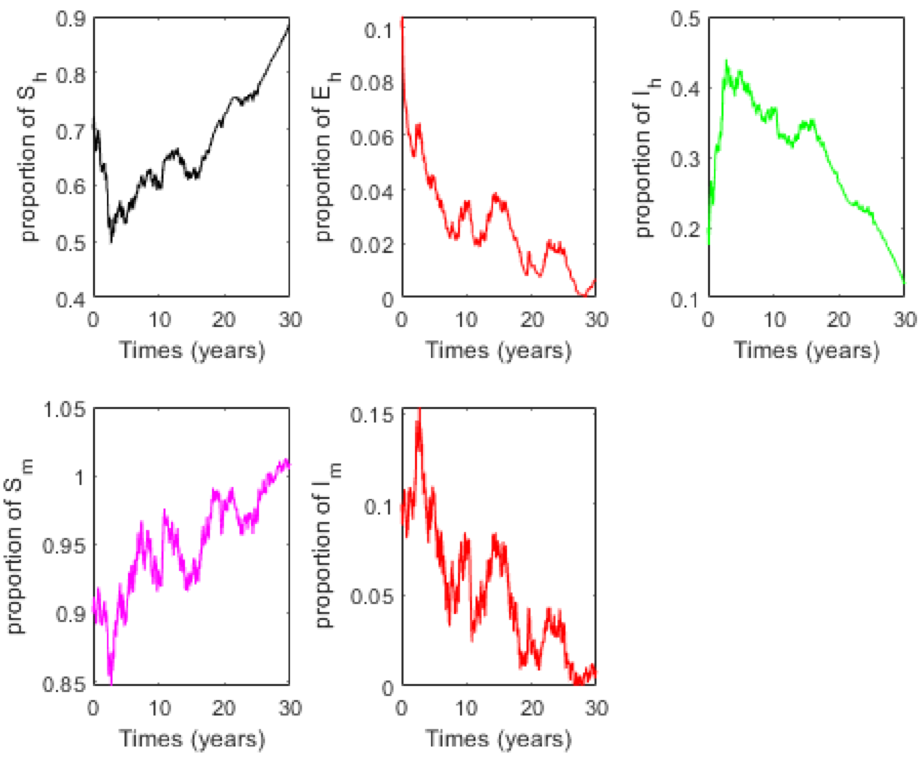

Figure 4: Graphs showing the transmission dynamics of filariasis for $\mathcal{R}_0^s = 0.68$.

5 Numerical example and remarks

Let's judiciously choose values for the parameters, Λ_h, Λ_m, β_h, β_m, μ_h, μ_m, θ_h and λ for which, the deterministic model is in extinction yet there is persistence in the mean for the stochastic model.
Consider the following table containing data when Lymphatic Filariosis is spreading:

Parameters	Λ_h	Λ_m	β_h	β_m	μ_h	μ_m	θ_h	λ	ϵ
Values	$0.00242.10^4$	$4.227.10^4$	0.015	0.91	0.8	3.623	0.25	0.01	0.00001

Thresholds	$\mathcal{R}_0$	$\mathcal{R}_0^s$
Values	0.068	0.89

We can notice that $\mathcal{R}_0$ is less than one thus according to the theorem 2.2 the equilibrium point E_0 of system (1) is globally asymptotically stable that is to say that the Lymphatic Filariosis stops spreading. However $\mathcal{R}_0^s$ being less than one shows that Lymphatic Filariosis persists in the mean according to the theorem 3.3 under the assumptions $(\mathcal{H})_1 - (\mathcal{H})_3$. Hence the importance to take the randomness aspect into account in the modelling of the spread of Lymphatic Filariosis.

6 Conclusion

In this paper, we focused on the comparative mathematical analysis of a deterministic and a stochastic epidemic model of Lymphatic filariasis lymphatic. First, we built a deterministic model of lymphatic filariasis. We showed the local stability of the disease-free equilibrium point by the Routh Hurtwiz criterion and then showed the global stability of this point by using the theorem of Varga and standard comparison theorem of Laksmikantham et al. Then we developed a stochastic model by adding white noises at the contact rates. This addition is done in order to take into account the fluctuations in the transmission of filariasis lymphatic. We have shown the existence and uniqueness of a positive solution using a Lyapunov function and the Itô's formula. To analyse the extinction of filariasis lymphatic, we established the almost certain convergence of the solution to disease-free equilibrium point when $\mathcal{R}_0$ is less than one, by successively constructing a Lyapunov function and applying the Itô's formula. We have also, established a persistence condition in mean of the stochastic differential system by constructing an appropriate Lyapunov function followed by an application of the Itô's formula and by using many other methods of stochastic analysis. In the last section we performed simulations to evaluate our results and then compared the two models. However, challenges remain in this work. We intend to conceive and analyse a discrete stochastic model of filariasis lymphatic. We also wish to analyse the transmission dynamics of other vector-borne diseases such as yellow fever and Zika.

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